Questions on Kinematics

1. **Annulas Mapping** Consider the annular 2-D body $\Omega = \{(X_1, X_2) \mid 1 < \sqrt{X_1^2 + X_2^2} < 2\}$. Draw the deformed body associated with the following motion.

$$
\begin{aligned}
u_r &= \boldsymbol{u} \cdot \boldsymbol{e}_r = 0.4 * (R-1)^2 \cos(3\Theta) \\
u_\theta &= \boldsymbol{u} \cdot \boldsymbol{e}_\theta = 0.4 * (R-1)^3
\end{aligned}
$$

Note:

1. $R = \sqrt{X_1^2 + X_2^2}$ and $\Theta = \tan^{-1}[X_2/X_1]$ are simply the polar coordinates of a point.
2. $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ are the unit orthonormal base vectors in polar coordinates. $\boldsymbol{e}_r = \cos(\theta)\boldsymbol{e}_1 + \sin(\theta)\boldsymbol{e}_2$ and $\boldsymbol{e}_\theta = -\sin(\theta)\boldsymbol{e}_1 + \cos(\theta)\boldsymbol{e}_2$.
3. Don't do this by hand! Use a computer.

2. **Rectangle Mapping** Given the 2-D body $\Omega = \{(X_1, X_2) \mid 0.1 < X_1 < 1\,, \quad 0.1 < X_2 < 1\}$ and the displacement field

$$
\begin{aligned}
u_1 &= \boldsymbol{u} \cdot \boldsymbol{e}_1 = 0.2 * \ln(1 + X_1 + X_2) \qquad (1) \\
u_2 &= \boldsymbol{u} \cdot \boldsymbol{e}_2 = 0.2 * \exp(X_1) \qquad (2)
\end{aligned}
$$

plot the displaced shape of the body.

Hand made plots are not acceptable for this question. Use a computer.

3. **Displacement Mapping** Given the 2-D body $\Omega = \{(X_1, X_2) \mid 0.1 < X_1 < 1\,, \quad 0.1 < X_2 < 1\}$ and the displacement field

$$
\begin{aligned}
u_r &= \boldsymbol{u} \cdot \boldsymbol{e}_r = 0.2 * \exp(X_1) \qquad (3) \\
u_\theta &= \boldsymbol{u} \cdot \boldsymbol{e}_\theta = 0.2 * \ln(1 + X_1 + X_2) \qquad (4)
\end{aligned}
$$

plot the displaced shape of the body.

4. **Deformed Curve** Consider a curve in an undeformed body that is given by $\boldsymbol{c}(s) \in \mathbb{R}^3$ for $s \in [0, 1]$; i.e. $\boldsymbol{c} : [0, 1] \to \mathbb{R}^3$. Recall that the expression for the length of this curve is given by $l(\boldsymbol{c}) = \int_0^1 \|\frac{d}{ds}\boldsymbol{c}(s)\|\, ds$. Assume

$$
\boldsymbol{c}(s) = \left[\frac{\sqrt{3}}{2}\cos(s\pi) + 1\right]\boldsymbol{e}_1 + \left[\frac{1}{2}\cos(s\pi) + 2\right]\boldsymbol{e}_2 + \sin(\pi s)\boldsymbol{e}_3
$$

(a) Determine the length of the curve.

(b) Assume the body is deformed by a pure shear motion such that $\boldsymbol{x} = \boldsymbol{X} + X_2\boldsymbol{e}_1$. Determine the length of the deformed curve. Express your answer to at least 3 significant digits. [Note that you may need to use an approximation technique to get your final answer.]

5. **Deformed Curve** The motion of a body is given by:

$$x_1 = 4X_1 - 3(X_2 - 2) \tag{8}$$
$$x_2 = X_1^3 \tag{9}$$
$$x_3 = X_3\,. \tag{10}$$

Consider the line joining the points $(1, 1, 0)$ and $(2, 2, 0)$ in the undeformed configuration of the body and compute its length in the deformed configuration. Note that the line becomes a curve when the motion occurs! You may evaluate any integrals using numerical integration; however, answers must be accurate to at least 3 significant digit.

6. **Deformed Curve** For the motion in Problem 25, consider a vertical line scored on the outer surface of the rod from bottom to top in the reference configuration. Assume that $\theta(X_3) = \alpha X_3$, where α is a given constant and determine the length of the "line" after the torsional motion is imposed on the rod. Note the "line" is now a curve. [Hint: Consider the original line as a bunch of short vectors and think about how each maps into the spatial configuration in order to generate a useful expression for finding the length in the deformed configuration.]

7. **Deformed Curve** Consider a 2-dimensional body $\mathcal{B} = \{(X_1, X_2) \mid 0 \leq X_1 \leq L \text{ and } -h/2 \leq X_2 \leq h/2\}$. The motion of the body is described by the mapping:

$$\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}) = \frac{L}{2\pi}\left(1 - \frac{3}{5h}X_2\right)\left\{\cos\left[\frac{2\pi X_1}{L}\right]\hat{\boldsymbol{e}}_1 + \sin\left[\frac{2\pi X_1}{L}\right]\hat{\boldsymbol{e}}_2\right\}\,.$$

Determine the length of the material line $\mathcal{C} = \{(X_1, X_2) \mid X_2 = h/2\}$ after deformation – i.e. the length of $\mathcal{C}_t$.

8. **Deformed Curve** Consider a round rod of radius R and length L. The rod has a helical scratch cut into its outer surface of the form $\boldsymbol{L}(s) = R\cos(2\pi s)\boldsymbol{e}_1 + R\sin(2\pi s)\boldsymbol{e}_2 + ps\boldsymbol{e}_3$ for $s \in [0, \frac{L}{p}]$, where the coordinate frame is centered at the base of the rod, with the three direction pointing along the rod's central axis, and p is a given parameter (the pitch of the helix).

   1. Show that the length of the helix is $L\sqrt{4\pi^2(R/p)^2 + 1}$ by integrating the norm of the curve's tangent vector, viz. $\int_0^{L/p} \|d\boldsymbol{L}/ds\|\, ds$.
   2. Assume the rod is now deformed by a homogenous deformation with deformation gradient $\boldsymbol{F} = (1/\sqrt{\lambda})\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + (1/\sqrt{\lambda})\boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \lambda\boldsymbol{e}_3 \otimes \boldsymbol{e}_3$, where λ is a given parameter. What is the new length of the helix?
   3. At what value of λ does the helix reach a minimal length?

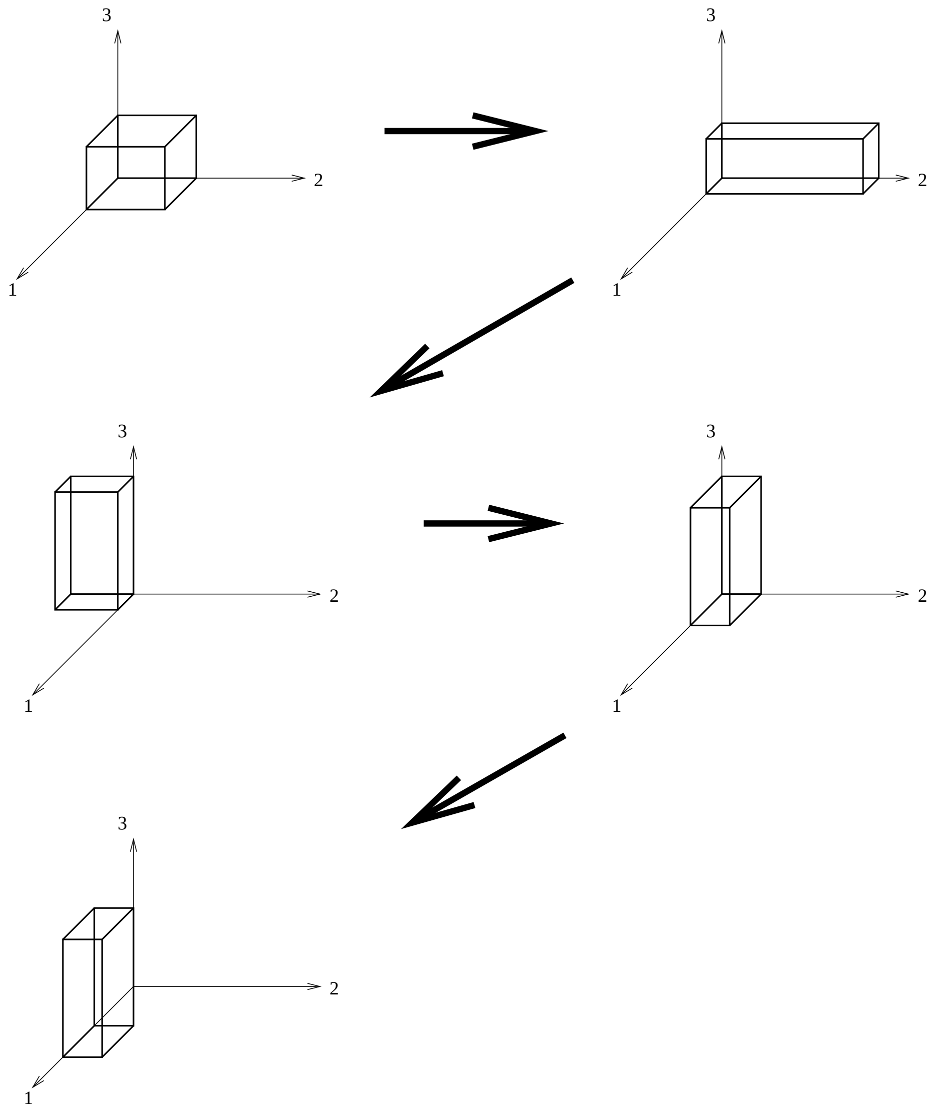
9. **Strain Measures** Let $\chi_i(\boldsymbol{X}) = KX_A\delta_{iA}$ where the constant $K \in \mathbb{R}$, $K > 0$. Find $\boldsymbol{F}$, $\boldsymbol{C}$, and $\boldsymbol{E}$.

10. **Analysis of Motion** For the deformation map $x_1 = X_1$, $x_2 = X_2 + kX_1^2$, $x_3 = X_3$, find:

1. The deformation gradient.
2. The stretch at $(1, 1, 1)$ in the direction oriented along the vector $(1, 2, 3)$. [Warning! Normalize!]
3. The strain at this point in the $(1, 0, 0)$ direction.
4. The orthogonal shear at this point between the two direction $(1/\sqrt{2}, 1/\sqrt{2}, 0)$ and $(-1/\sqrt{2}, 1/\sqrt{2}, 0)$.

11. **Composite Motion** Consider the homogeneous deformation of a cube with side length a which is composed: first of stretching in the 1-, 2-, and 3-directions of magnitudes 1, 5, and 2, respectively; followed by a $+\pi/2$ rotation about the 1-axis; followed by a $+\pi/2$ rotation about the 3-axis; and then a rigid translation in the 1-direction of magnitude a. (See the accompanying figure).

    1. What is the deformation gradient for this motion?
    2. What happens to the (material) tangent vectors which were aligned in the $\boldsymbol{e}_i$ directions after deformation?
    3. What is the deformation mapping for this motion?



12. **Almansi Strain Tensor** Show

$$d\boldsymbol{X} \cdot d\boldsymbol{X} - d\boldsymbol{x} \cdot d\boldsymbol{x} = d\boldsymbol{x} \cdot 2\boldsymbol{e} d\boldsymbol{x} \tag{11}$$

where $\boldsymbol{e} = \frac{1}{2}(\boldsymbol{b}^{-1} - \mathbf{1})$ and $\boldsymbol{b} = \boldsymbol{F}\boldsymbol{F}^T$. Also show

$$\boldsymbol{e} = \frac{1}{2}\left(-\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} - \left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}\right)^T + \left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}\right)^T \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}\right) \tag{12}$$

Note that $\boldsymbol{X} = \boldsymbol{x} - \boldsymbol{u}$.

13. **Correspondence between eigenstructure of $\boldsymbol{b}$ and $\boldsymbol{C}$** Consider the left Cauchy-Green deformation tensor $\boldsymbol{b} = \boldsymbol{F}\boldsymbol{F}^T$.

(a) Show that $\boldsymbol{b}$ has the same eigenvalues as the right Cauchy-Green deformation tensor $\boldsymbol{C}$.

(b) Express the eigenvectors of $\boldsymbol{b}$ in terms of those of $\boldsymbol{C}$.

14. **Correspondence between eigenstructure of $\boldsymbol{C}$ and $\boldsymbol{E}$** Prove that the Green-Lagrange strain tensor, $\boldsymbol{E}$, and the right Cauchy-Green strain tensor, $\boldsymbol{C}$, have the same eigenvectors. Find the relationship between the eigenvalues of $\boldsymbol{E}$ and $\boldsymbol{C}$.

15. **Adjugate Derivation** Prove $J\boldsymbol{F}^{-T}(\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{F}\boldsymbol{A} \times \boldsymbol{F}\boldsymbol{B}$.

16. **Torsional Motion** The torsional motion of a right circular cylinder can be approximated as

$$\boldsymbol{\phi} \rightarrow \begin{pmatrix} X_1 \cos(\beta X_3) + X_2 \sin(\beta X_3) \\ -X_1 \sin(\beta X_3) + X_2 \cos(\beta X_3) \\ X_3 \end{pmatrix} \tag{25}$$

where X_3 is the axial coordinate of the cylinder and β is the twist rate of the cylinder (a constant). Find $\boldsymbol{F}$, $\boldsymbol{C}$, and $\boldsymbol{E}$.

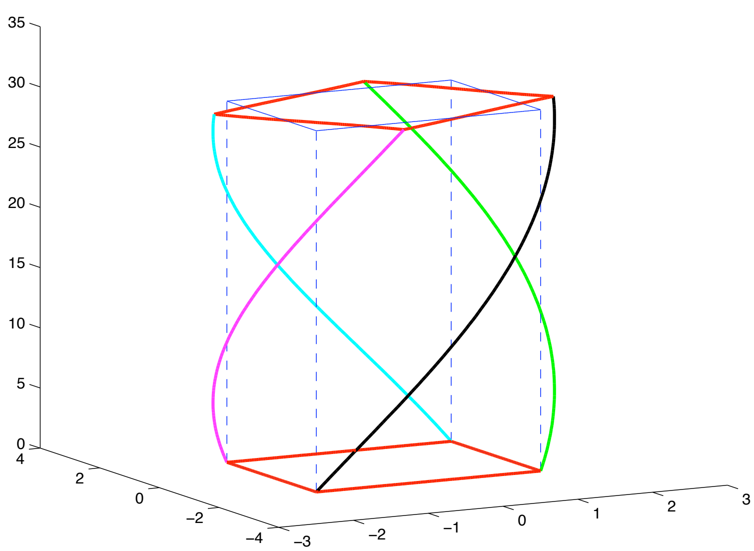
17. **Torsional Motion** Consider a square bar with side lengths 3 cm and length 30 cm in the reference configuration; i.e. $\mathcal{R} = \{\boldsymbol{X} \mid (X_1, X_2, X_3) \in [-1.5, 1.5] \times [-1.5, 1.5] \times [0, 30]\}$. The bar undergoes a twisting deformation where the bottom is fixed and the top rotates by an angle Θ over the range $[0, 2\pi]$. Assume, for the purposes of this assignment, that the deformation map is given (in consistent units) as:

$$x_1 = X_1 \cos(\Theta X_3/30) - X_2 \sin(\Theta X_3/30) \tag{29}$$

$$x_2 = X_1 \sin(\Theta X_3/30) + X_2 \cos(\Theta X_3/30) \tag{30}$$

$$x_3 = X_3 + \frac{\Theta}{30}(1 - \exp(-X_3/3))[X_1 X_2 - \frac{72}{\pi^3 \cosh(\pi/2)} \sin(\pi X_1/3) \sinh(\pi X_2/3)] \tag{31}$$

The motion for $\Theta = 2\pi/3$ looks like:



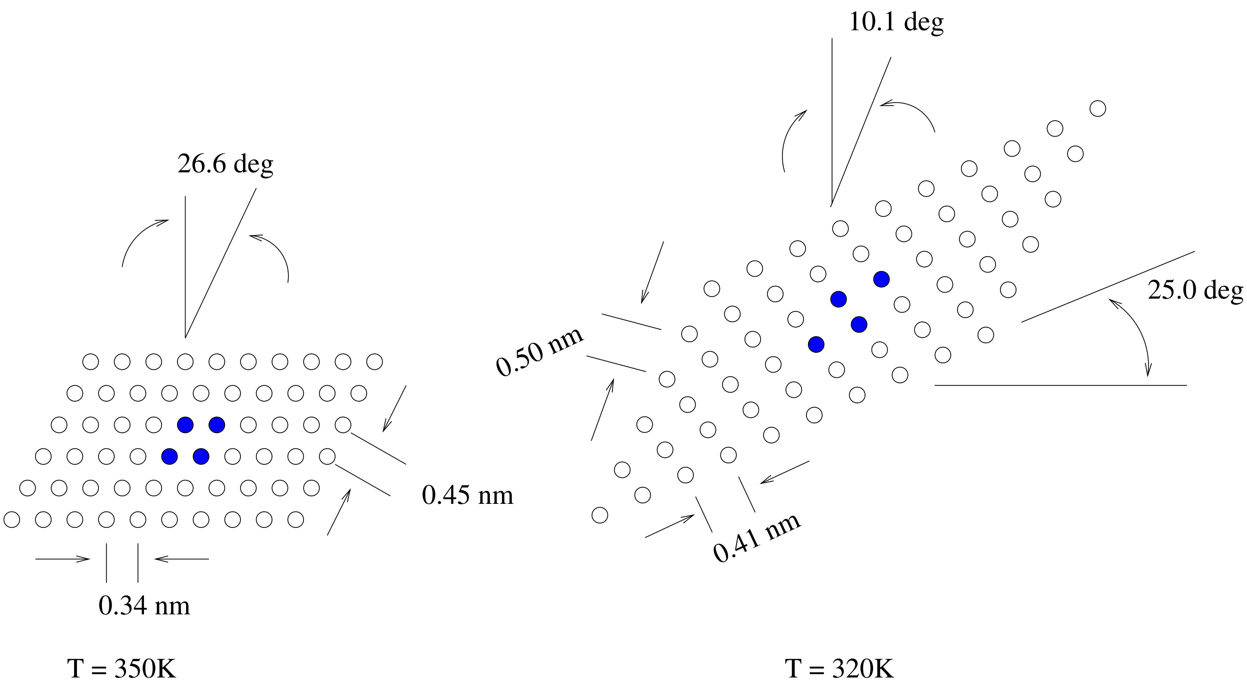
   1. Compute the deformation gradient.
   2. Consider the point $\boldsymbol{X} = (1.5, 0, 15)$ and a rotation $\Theta = 2\pi/3$

(a) What are the components of $\boldsymbol{F}$ at this point?
(b) What are the components of $\boldsymbol{C}$ at this point?
(c) What are the components of $\boldsymbol{U}$ at this point?
(d) What are the components of $\boldsymbol{R}$ at this point?
(e) Consider a triad of local vectors at this point in the three coordinate directions. In which direction to they point after deformation?
(f) What is the maximum stretch at this point? and in what direction does it occur?
(g) What is the maximum (elongational) strain at this point? and in what direction does it occur?
(h) What is the volume strain at this point?
(i) Compute the orthogonal shear strain at this point with respect to the 1 and 2 directions, the 1 and 3 directions, and the 2 and 3 directions.

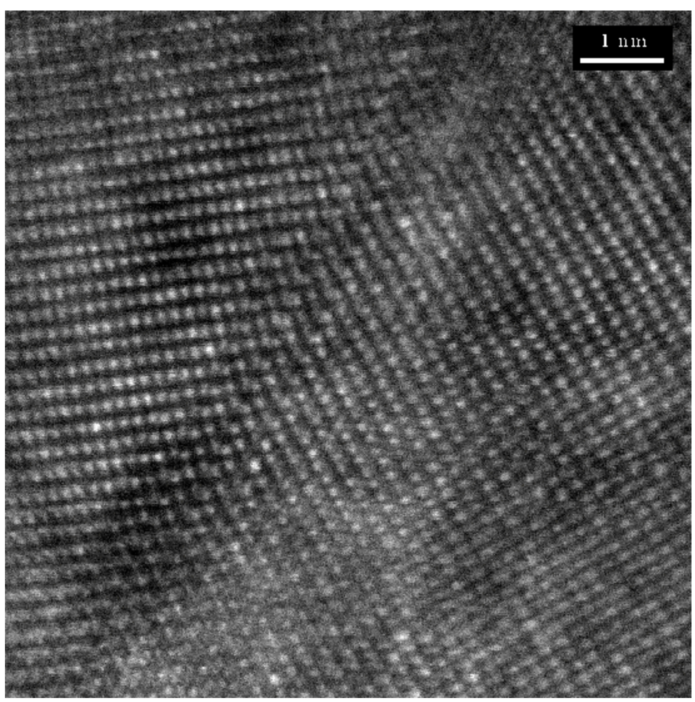
3. Consider the same point and rotation magnitude as in Part 2. The point sits on the surface of the bar and the unit outward normal is $\boldsymbol{n} = \boldsymbol{e}_1$. Consider a small square area of material centered at this point on the surface of dimension 0.1×0.1.
   (a) What is the magnitude of this local area after deformation?
   (b) What is the normal vector to this area after deformation?

4. (Extra) Consider the edge $\mathcal{C} = \{\boldsymbol{X} \mid X_1 = 1.5 \text{ and } X_2 = 1.5\}$.
   (a) What is length of $\mathcal{C}$ (before deformation)?
   (b) Plot the length of $\mathcal{C}_t$ as a function of $\Theta \in [0, 2\pi]$. (Hint: Compute the length using numerical quadrature; i.e. think about breaking up the edge into a collection of short vector, computing the lengths of the short vectors after deformation, and then adding up these lengths to computer the length of the edge. Two digits of accuracy is more than sufficient for this question.)

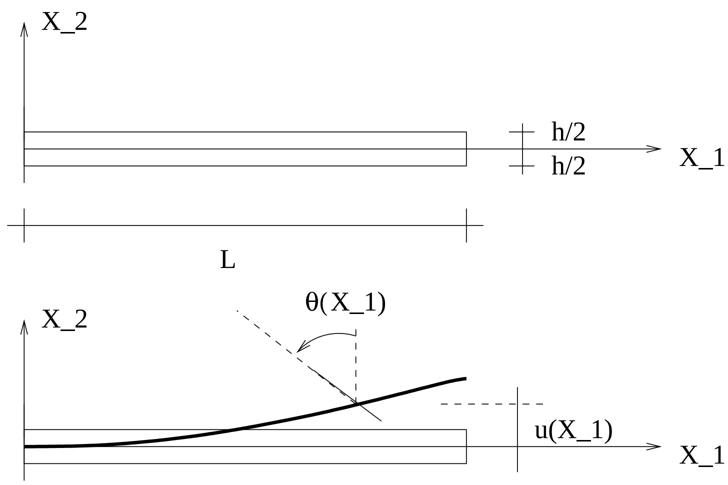
18. **Transformation Stretch: High Res TEM** High resolution transmission electron microscopy (HRTEM) is an experimental method that allows one to image materials down to sub-atomic level resolution. When applied to crystalline materials (say metals) it allows one to image the location of the atoms accurately. Consider the two schematic representations of what one can typically find when applied to a metal sample at two different temperatures.



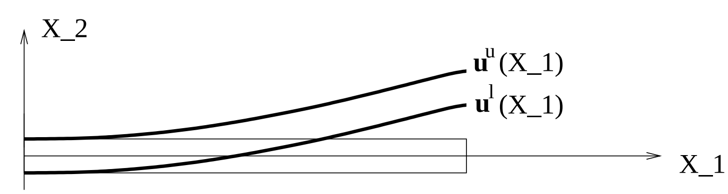
Upon cooling from 350 K to 320 K the material has undergone a martensitic (diffusionless) transformation from one set of atomic spacings and angles to another. Determine the "transformation" stretch-tensor associated with the transformation shown. Treat as a two-dimensional problem; note, the diagram is not to scale. Below however is an actual HRTEM image just so you can have an appreciation of what is experimentally possible. The white dots are the atoms! The scale bar at the top right is only 1 nm.



19. Timoschenko Beam: Alternate Form The classical characterization of a shear deformable beam is given by two scalar-valued functions $u(X_1)$ and $\theta(X_1)$ of axial position which represent the vertical displacement of points along the neutral axis and rotation of the cross-section relative to the vertical, respectively. It is assumed that plane sections remain plane. An alternative characterization of the motion can be achieved by considering two vector-valued functions of axial position, $\boldsymbol{u}^u(X_1)$ and $\boldsymbol{u}^l(X_1)$, which represent the displacement of the upper and lower chords of the beam (still assuming the cross-section remains planar).



[Standard Beam Motion Characterization]



[Alternative Beam Motion Characterization]

Using this alternative characterization, compute expressions for:

1. $\boldsymbol{\chi}(\boldsymbol{X})$, the deformation map.
2. $\boldsymbol{F}$, the deformation gradient.
3. E_{11}, the 11 component of the Green-Lagrange Strain tensor.

[Note: This is a two dimensional problem.]

20. **Shear Deformable Beam** Consider the deflection of beam in terms of $u(X_1)$ the vertical motion of the neutral axis and $\theta(X_1)$ the rotation of the vertical fibers. Let the length of the beam be L and the depth of the beam be h, where $L/h = 5$. Further assume that $u(X_1) = L\exp[X_1/L]$ and $\theta(X_1) = 1 + (X_1/L)$.

1. Compute and accurately plot the normal strain H along the top fiber of the beam; i.e. plot $H((X_1, h/2), \boldsymbol{e}_1)$.
2. Compute and accurately plot the orthogonal shear strain along the neutral axis between $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$.

21. **Beam Stretch Tensor and Rotation** Consider a beam with reference configuration $\mathcal{R} = \{\boldsymbol{X} \mid X_1 \in [0, L] \text{ and } X_2 \in [-\frac{h}{2}, \frac{h}{2}]\}$ and a deformation:

$$\begin{aligned} x_1 &= X_1 - X_2 \sin[\theta(X_1)] \\ x_2 &= X_2 + u(X_1) - X_2(1 - \cos[\theta(X_1)]), . \end{aligned}$$

Assume $\theta(X_1) = X_1/L$ and $u(X_1) = \frac{L}{2}(X_1/L)^2$. Determine the value of the right stretch tensor field and rotation tensor field at the tip of the beam $(X_1, X_2) = (L, 0)$. Compute the tensors numerically (not analytically). Use a calculator capable of eigencomputations or, better, use MATLAB.

22. **Area Change** The motion of a body is given by:

$$\begin{aligned} x_1 &= 4X_1 - 3(X_2 - 2) \\ x_2 &= (X_1)^3 \\ x_3 &= X_3 \,. \end{aligned}$$

Consider the square area whose 4 corners are the points $(1, 1, 0)$, $(1, 2, 0)$,$(2, 2, 0)$, and $(2, 1, 0)$ in the undeformed configuration of the body. After deformation the square changes shape to something that is no longer a square. Compute the area of this new shape. Note that the sides of the square becomes curves when the motion occurs! Hints: (1) This is a finite deformation problem. (2) Nanson's formula $\boldsymbol{n} \mathrm{d}a = J\boldsymbol{F}^{-T}\boldsymbol{n}_R \mathrm{d}A$.

23. **Area Strain** Consider a thin square sheet which occupies a region $[-\frac{L}{2}, \frac{L}{2}] \times [-\frac{L}{2}, \frac{L}{2}] \times [-\frac{t}{2}, \frac{t}{2}]$, where $t \ll L$. The sheet is deflected such that its deformation map is given by:

$$\begin{aligned} x_1 &= X_1 \\ x_2 &= X_2 \\ x_3 &= X_3 + k\left(X_1 - \frac{L}{2}\right)^2 \left(X_2 - \frac{L}{2}\right)^2 , \end{aligned}$$

where k is a given constant. Consider a small patch of material on the surface of the sheet near the point $(\frac{L}{4}, \frac{L}{4}, \frac{t}{2})$ and determine area strain at this point; i.e. determine $\Delta A/A$ at this point for the material on the outer surface of the body.

24. **Screw Dislocation** A screw dislocation in a solid is characterized by a displacement field of the form

$$u_3(\boldsymbol{X}) = \frac{b}{2\pi} \tan^{-1}\left(\frac{X_1}{X_2}\right) ,$$

where b is the (given) magnitude of the Burger's vector and the dislocation is assumed to align with the 3-axis; $u_1 = u_2 = 0$. Find the Green-Lagrange strain tensor associated with the dislocation.

25. **Torsional Volume Strain** Consider a round rod of radius R and length L with reference placement $\mathcal{R} = \{\boldsymbol{X} \mid (X_1)^2 + (X_2)^2 \leq R^2 \text{ and } 0 \leq X_3 \leq L\}$. The rod undergoes a torsional motion:

$$\begin{aligned} x_1 &= X_1 \cos(\theta(X_3)) + X_2 \sin(\theta(X_3)) \\ x_2 &= -X_1 \sin(\theta(X_3)) + X_2 \cos(\theta(X_3)) \\ x_3 &= X_3 \end{aligned}$$

where $\theta(X_3)$ is a given but unspecified function. Compute the volumetric strain field.

26. **Rigid Body Mechanics** Consider a body $\mathcal{B}$ undergoing a time dependent rigid motion $\boldsymbol{x}(t) = \boldsymbol{R}(t)\boldsymbol{X} + \boldsymbol{c}(t)$, where $\boldsymbol{R}(t) \in SO(3)$ and $\boldsymbol{c}(t) \in \mathbb{R}^3$ are known.

    1. Show that the velocity $\boldsymbol{v} = \dot{\boldsymbol{x}}$ can be written as $\boldsymbol{v} = \boldsymbol{\omega} \times (\boldsymbol{x} - \boldsymbol{c}) + \dot{\boldsymbol{c}}$, where $\boldsymbol{\omega}$ is a suitably defined vector. Hint, $\dot{\boldsymbol{R}}\boldsymbol{R}^T \in so(3)$; i.e. it is skew-symmetric.
    2. Define the center of mass position of the body as

    $$\bar{\boldsymbol{x}} = \frac{1}{M} \int_{\mathcal{B}_t} \boldsymbol{x} \rho dv$$

    and the center of mass velocity as

    $$\bar{\boldsymbol{v}} = \frac{1}{M} \int_{\mathcal{B}_t} \boldsymbol{v} \rho dv \, .$$

    Show the center of mass velocity can be written as $\bar{\boldsymbol{v}} = \boldsymbol{\omega} \times (\bar{\boldsymbol{x}} - \boldsymbol{c}) + \dot{\boldsymbol{c}}$. It directly follows, then that the linear momentum of the body $\boldsymbol{l} = \int_{\mathcal{B}_t} \rho \boldsymbol{v} dv$ can be written as $\boldsymbol{l} = M\bar{\boldsymbol{v}}$, where $M = \int_{\mathcal{B}_t} \rho dv$ is the mass of the body.
    3. The angular momentum of the body is defined to be $\boldsymbol{h} = \int_{\mathcal{B}_t} \boldsymbol{x} \times \rho \boldsymbol{v} dv$. Show that this can be written as

    $$\boldsymbol{h} = \bar{\boldsymbol{x}} \times \boldsymbol{l} + \mathbb{J}\boldsymbol{\omega} \, ,$$

    where the second order inertia tensor $\mathbb{J} = \int_{\mathcal{B}_t} \rho[(\boldsymbol{x}-\bar{\boldsymbol{x}})\cdot(\boldsymbol{x}-\bar{\boldsymbol{x}})\mathbf{1}-(\boldsymbol{x}-\bar{\boldsymbol{x}})\otimes(\boldsymbol{x}-\bar{\boldsymbol{x}})]dv$. Hint, it the last step it is useful to note that $\boldsymbol{a} \times \boldsymbol{b} \times \boldsymbol{c} = [(\boldsymbol{a} \cdot \boldsymbol{c})\mathbf{1} - \boldsymbol{c} \otimes \boldsymbol{a}]\boldsymbol{b}$. To use this hint, however, you must first prove it.

27. **Speckle Field Interferometry** Speckle pattern interferometry is an experimental methodology which provides near full field two dimensional deformation mapping information for the surface of a body. The accompanying file provides data in the following format: in each row one finds in columns 1 and 2 the X_1 and X_2(in-plane) reference coordinates of a material point and in columns 3 and 4 the corresponding displacement components u_1 and u_2 of the same material point. The data is representative of a 50 by 50 grid of data points in the center of the measurement field. Estimate

    1. the magnitude,

2. material direction and
3. material location of the maximum stretch and
4. the magnitude and
5. material location of the maximum volumetric strain.

Assume that there is no displacement in the out-of-plane direction.

28. **Incompressible Measures** Let $\boldsymbol{\varphi} : \Omega \rightarrow \mathbb{R}^3$ be a given motion and let $\boldsymbol{F} = \partial \boldsymbol{\varphi} / \partial \boldsymbol{X}$ be the deformation gradient. Define

$$\bar{\boldsymbol{F}} = J^{-\frac{1}{3}} \boldsymbol{F}$$

where $J = \det[\boldsymbol{F}]$.

(a) Justify the name *volume preserving part of* $\boldsymbol{F}$ often assigned to $\bar{\boldsymbol{F}}$. Give a physical interpretation.

(b) Define:

$$\bar{\boldsymbol{l}} = (\dot{\bar{\boldsymbol{F}}} \bar{\boldsymbol{F}}^{-1}) ; \quad \bar{\boldsymbol{d}} = \frac{1}{2} (\bar{\boldsymbol{l}} + \bar{\boldsymbol{l}}^T) \tag{2}$$

Find the relation between $\bar{\boldsymbol{l}}$ and $\boldsymbol{l} = \partial \boldsymbol{v} / \partial \boldsymbol{x}$, where $\boldsymbol{v}(\boldsymbol{x})$ is the spatial velocity field. Find the relation between $\boldsymbol{d}$ and $\bar{\boldsymbol{d}}$. Compute $\text{tr}[\bar{\boldsymbol{d}}]$ and $\text{tr}[\bar{\boldsymbol{l}}]$. Give a physical interpretation. [Remark: These relations play a crucial role in the mechanics of incompressible materials and large deformation plasticity.]

29. **Function Linearization** Consider the vector valued function $\boldsymbol{v}(\boldsymbol{H}) = \boldsymbol{H} \cdot \boldsymbol{H} \cdot \boldsymbol{a} + (\boldsymbol{1} : \boldsymbol{H}) \boldsymbol{a}$, where $\boldsymbol{H}$ is a second order tensor and $\boldsymbol{a}$ is a given constant vector. Linearize the function $\boldsymbol{v}(\boldsymbol{H})$ about $\boldsymbol{H}_o = \boldsymbol{1}$; i.e. find a linear approximation to $\boldsymbol{v}(\boldsymbol{H})$ for values of $\boldsymbol{H}$ near $\boldsymbol{1}$.

30. **Function Linearization** Consider the function $f(\boldsymbol{H}) = \cos(\boldsymbol{H} : \boldsymbol{H})$ where $\boldsymbol{H}$ is the gradient of the displacement field. Find the linearization of f near $\boldsymbol{H} = \boldsymbol{0}$.

31. **Linearization Mooney-Rivlin Energy** Linearize the function $f(\boldsymbol{H}) = k_1(I_1 - 3) + k_2(I_2 - 3)$ about $\boldsymbol{H} = \boldsymbol{0}$, where k_1 and k_2 are given constants and I_1 and I_2 are, respectively, the first and second invariants of $\boldsymbol{H}$.

32. **Linearization of Rotation** For every $\boldsymbol{R} \in SO(3)$ there exists three orthonormal vectors $\{\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}\}$ and a scalar $\theta \in \mathbb{R}$ such that

$$\boldsymbol{R} = \boldsymbol{p} \otimes \boldsymbol{p} + \cos(\theta)[\boldsymbol{q} \otimes \boldsymbol{q} + \boldsymbol{r} \otimes \boldsymbol{r}] - \sin(\theta)[\boldsymbol{q} \otimes \boldsymbol{r} - \boldsymbol{r} \otimes \boldsymbol{q}] .$$

Linearize this expression for small angles of rotation θ and show that such rotations can be approximated by the identity plus a skew-symmetric tensor.

33. **Spin Tensor** Consider a time dependent tensor $\boldsymbol{Q}(t) \in SO(3)$ – i.e. a rotation tensor. Show that

1. the tensor $\boldsymbol{\Omega}(t) = \dot{\boldsymbol{Q}} \cdot \boldsymbol{Q}^T$ is skew-symmetric (i.e. $\boldsymbol{\Omega} \in so(3)$) and
2. if $\boldsymbol{a}(t) = \boldsymbol{Q}(t) \cdot \boldsymbol{A}$, then $\dot{\boldsymbol{a}}(t) = \boldsymbol{\Omega} \cdot \boldsymbol{a}$, where $\boldsymbol{A}$ is a given constant vector.

34. **Linearized Jacobian** Consider a displacement field is $\boldsymbol{u} = [20X^2Y\boldsymbol{e}_1 + 10(Y^2+Z^2)\boldsymbol{e}_2 + (X+3Z^3)\boldsymbol{e}_3] \times 10^{-2}$. Find the deformation gradient, the Jacobian, and small deformation volume strain. Assess the the approximation $\text{Lin}[J] = 1 + u_{i,i}$ at the point $(1, 2, -3)$. $[(X, Y, Z) = (X_1, X_2, X_3)]$

35. **Analysis of Motion** Consider the deformation map $x_1 = X_1$, $x_2 = X_2 + kX_1^2$, $x_3 = X_3$ for a tri-unit cube $\mathcal{R} = [0, 1] \times [0, 1] \times [0, 1]$, where k is a scalar parameter.
    1. Compute the normal strain at $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ in the direction oriented along the vector $(1, 2, 3)$. [Warning! Normalize!]
    2. Compute the orthogonal shear at this point between the two direction $(1/\sqrt{2}, 1/\sqrt{2}, 0)$ and $(-1/\sqrt{2}, 1/\sqrt{2}, 0)$.
    3. For parts 1 and 2, determine the value below which k has to be for the small strain approximations to have a relative error of less than 10^{-4}. For the case of orthogonal shear, also examine the absolute error.

36. **Deformation of a bar:** Consider a square bar $\Omega = [-\frac{a}{2}, \frac{a}{2}] \times [-\frac{a}{2}, \frac{a}{2}] \times [0, L]$ – side length a and longitudinal length L. The bar is deformed according to

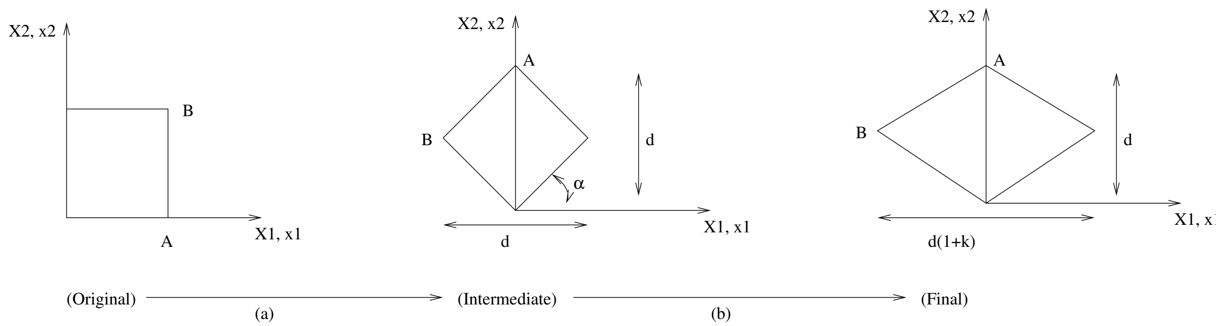
$$\boldsymbol{\varphi}(\boldsymbol{X}) = (X_1 + \beta X_2 + \alpha X_3)\, \boldsymbol{e}_1 + (X_2 - \beta X_1 + \alpha X_3)\, \boldsymbol{e}_2 + \left(X_3 + \alpha X_3^2\right) \boldsymbol{e}_3\,, \qquad (45)$$

where α and β are given constants.
    1. Determine the deformation gradient; assume arbitrary α and β.
    2. Determine the Green-Lagrange strain tensor; assume arbitrary α and β.
    3. Under the assumption that $|\alpha| \ll 1$ and that $|\beta| \ll 1$:
        (a) Determine the small strain tensor.
        (b) Assuming a linear elastic isotropic material, determine the Cauchy stress tensor.
        (c) Determine the total force and moment/torque that must have been applied on the end of the bar, $X_3 = L$.

37. **Rotation of a Plate** A thin square plate underwent a deformation consisting of two subsequent processes, schematically shown in the figure.

    (a) a rotation by $\alpha = \pi/4$ (about the x_3 axis).

    (b) elongation in the x_1 direction, such that the ratio of the length of the fibers parallel to the x_1 axis is $d = 1 + k$, where k is a very small number ($k << 1$).



Calculate the following field quantities:

(i) total displacement (from the original state to the final state) as a function of X_1 and X_2.

(ii) components of the Green-Lagrange strain tensor $\boldsymbol{E}$ and its linear part $\boldsymbol{\varepsilon}$. Is $\boldsymbol{\varepsilon}$ a good approximation to $\boldsymbol{E}$? Comment on it.

38. **Validity of Small Strain Tensor** Consider a cube, in its undeformed configuration, that occupies the region $\Omega = [0,1] \times [0,2] \times [0,2]$. The cube is subject to a set of loads that results in a deformation map

$$\boldsymbol{\chi}(\boldsymbol{X}) = \delta(X_1 + X_2)\boldsymbol{e}_1 + \delta(X_3^2)\boldsymbol{e}_2 + \delta(X_1 X_2^2)\boldsymbol{e}_3 \,.$$

How small must δ be for the linear strain tensor, $\boldsymbol{\varepsilon}$, to be a valid approximation to the Green-Lagrange strain tensor, $\boldsymbol{E}$. Valid is defined for the purposes of this problem as

$$\max_{\boldsymbol{X} \in \Omega} \|\boldsymbol{\varepsilon}(\boldsymbol{X}) - \boldsymbol{E}(\boldsymbol{X})\| < 10^{-3}$$

Note that an appropriate definition for the norm of a tensor is $\|\boldsymbol{T}\| = \sqrt{\boldsymbol{T} : \boldsymbol{T}} = \sqrt{T_{ij} T_{ij}}$. A small computer program is perhaps the best way to solve this problem.

39. **Rigid Motion** Consider a rigid rotation $\boldsymbol{x} = \boldsymbol{R}(\boldsymbol{X} - \boldsymbol{0}) + \boldsymbol{c}$, where $\boldsymbol{R} \in SO(2)$ is constant and $\boldsymbol{c} \in \mathbb{R}^2$ is also constant. Noting that in component form we can express

$$R_{iA} = \begin{bmatrix} \cos(\beta) & \sin(\beta) \\ -\sin(\beta) & \cos(\beta) \end{bmatrix} , \qquad (46)$$

find $\boldsymbol{E}$ and compare it to $\boldsymbol{\varepsilon}$. Comment on the validity of $\boldsymbol{\varepsilon}$.

40. **Rigid Motion** Consider a rigid rotation $\boldsymbol{x} = \boldsymbol{Q} \cdot (\boldsymbol{X} - \boldsymbol{0}) + \boldsymbol{c}$, where $\boldsymbol{Q} \in SO(3)$ is a constant tensor and $\boldsymbol{c} \in \mathbb{R}^3$ is a constant vector. Noting that in component form we can in general write, for some coordinate frame,

$$Q_{iA} = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} ,$$

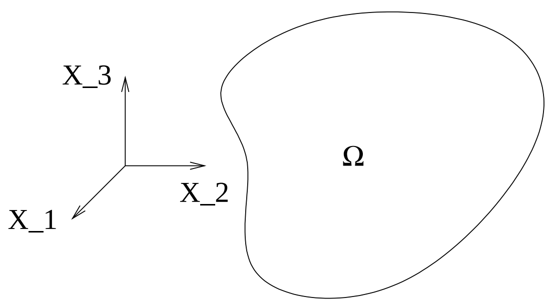
where $\alpha \in \mathbb{R}$ is the rotation angle.

1. Compute $\boldsymbol{E}$ and compare it to $\boldsymbol{\varepsilon}$.
2. Compute the rotation in the polar decomposition ($\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U}$)and compare it to $\mathbf{1} + \boldsymbol{\omega}$.

Comment on the validity of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\omega}$.

41. **Classical Derivation of Shear Formula** Derive γ_{12} via the classical construction of angle changes associated with neighboring points in the small strain setting.

42. **Incompressibility** An incompressible material is one where the volumetric strains must be equal to zero at all points. Consider an incompressible body Ω which is deformed by imposing displacements at all points on the boundary $\partial\Omega$. What condition must the imposed boundary displacements satisfy in order to be compatible with the incompressibility constraint? [Assume small displacement theory.]



43. **Small Deformation Timoshenko Beam** Consider a Timoshenko beam initially occupying the region $[0, L] \times [-h/2, h/2]$. The beam is deformed into a sinusoidal shape such that $u(X_1) = A\sin(kX_1)$ and $\theta(X_1) = 0$, where k and A are given constants. Assume small strains and compute the (orthogonal) shear strain with respect to $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ in the beam as a function of position.

44. **Small strain field analysis** Consider a (small) strain field $\boldsymbol{\varepsilon}(\boldsymbol{x})$ whose compents are given by

$$\varepsilon_{ij} = \begin{bmatrix} 3x & 5y+6z & z^3 \\ 5y+6z & 0 & x^2+y^2 \\ z^3 & x^2+y^2 & \exp(x) \end{bmatrix} \times 10^{-6}\,. \tag{54}$$

1. Find the principal strains and directions at $x_i = (1, 2, 3)$.
2. What is the normal strain in the direction $n_i = (1, 1, 1)$ at the point $x_i = (2, 2, 0)$?
3. What is the change in angle between $v_i^{(1)} = (1, 1, 1)$ and $v_i^{(2)} = (2, 1, 3)$ at the point $x_i = (1, 1, 1)$?
4. What is the volumetric strain at $x_i = (0, 0, 0)$?

Hint: don't waste your time doing these by hand! Use MATLAB or something similar.

45. **Small Strain Field Analysis** Consider a (small) strain field $\boldsymbol{\varepsilon}(\boldsymbol{x})$ whose components are given by

$$\varepsilon_{ij} = \begin{bmatrix} 3x_2x_2 & x_2+x_3 & (x_3)^3 \\ x_2+x_3 & x_1 & x_1+x_3 \\ (x_3)^3 & x_1+x_3 & \cos(x_1) \end{bmatrix} \times 10^{-6}\,. \tag{55}$$

   1. Find the principal strains and directions at $x_i = (2, 0, 1)$.
   2. What is the normal strain in the direction $n_i = (1, 1, 1)$ at the point $x_i = (2, 2, 0)$? [Careful: remember to normalize.]
   3. What is the change in angle between $v_i^{(1)} = (2, 1, 1)$ and $v_i^{(2)} = (2, 1, 2)$ at the point $x_i = (0, 1, 1)$?
   4. What is the volumetric strain at $x_i = (1, 0, 0)$?

Hint: Don't waste your time doing these by hand! Use MATLAB or something similar.

46. **Small Strain Field Analysis** The strain at a particular point in a body is given by

$$\boldsymbol{\varepsilon} \sim \begin{bmatrix} 7 & 8 & 0 \\ 8 & 9 & 3 \\ 0 & 3 & 55 \end{bmatrix} \times 10^{-5}$$

in the $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ basis where $\boldsymbol{a} = \frac{1}{\sqrt{3}}(\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3)$, $\boldsymbol{b} = \frac{1}{\sqrt{2}}(\boldsymbol{e}_1 - \boldsymbol{e}_2)$, and $\boldsymbol{c} = \frac{1}{\sqrt{6}}(\boldsymbol{e}_1 + \boldsymbol{e}_2 - 2\boldsymbol{e}_3)$.

   1. Find the max normal and shear strains at this point.
   2. Find the normal strain in the $\boldsymbol{e}_1$ direction at this point.
   3. Find the angle change between the $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ directions at this point.

47. **Small Strain Field Analysis** Let $\boldsymbol{a} = 1\boldsymbol{e}_1 + 1\boldsymbol{e}_2 + 1\boldsymbol{e}_3$, $\boldsymbol{b} = 2\boldsymbol{e}_1 + 1\boldsymbol{e}_2 + 1\boldsymbol{e}_3$. The homogeneous strain in a body is given as $\boldsymbol{\varepsilon} = (\boldsymbol{a} \cdot \boldsymbol{b})(\boldsymbol{a} \otimes \boldsymbol{b} + \boldsymbol{b} \otimes \boldsymbol{a})$.

   1. What is the minimum normal strain (in absolute value) and in which direction does it occur?
   2. Find any two vectors such that the angle between them does not change under the deformation.
   3. Now consider imposing an additional homogenous strain $\bar{\boldsymbol{\varepsilon}}$ on the body. What property must $\bar{\boldsymbol{\varepsilon}}$ have in order for the overall motion to be volume preserving?

48. **Wave Motion** Consider the following motion

$$\boldsymbol{x} = \boldsymbol{X} + \boldsymbol{b}\cos(\boldsymbol{k}\cdot\boldsymbol{X} - ct) \qquad t \geq 0$$

where $\boldsymbol{b}$ is a known constant vector such that $\|\boldsymbol{b}\| << 1$, $\boldsymbol{k}$ is a known constant vector, $c > 0$ is a known constant scalar, and t is time.

1. Determine the strain field associated with this motion.
2. Find the directions in which the normal strains must be zero for all times and at all points.
3. Find a condition between $\boldsymbol{k}$ and $\boldsymbol{b}$ such that the motion is isochoric at all points for all times.
4. Describe the motion physically.

49. **Spectral Representation and Transformation** You are told that the principal strains at a point are $(9.6235, 0.0000, -0.6235,) \times 10^{-6}$ and that the corresponding eigenvectors in the $\{\boldsymbol{e}_i\}$ basis are

$$\boldsymbol{n}^{(1)} \rightarrow \begin{pmatrix} 0.3851 \\ 0.5595 \\ 0.7339 \end{pmatrix}, \quad \boldsymbol{n}^{(2)} \rightarrow \begin{pmatrix} 0.4082 \\ -0.8165 \\ 0.4082 \end{pmatrix}, \quad \boldsymbol{n}^{(3)} \rightarrow \begin{pmatrix} 0.8277 \\ 0.1424 \\ -0.5428 \end{pmatrix}.$$

1. What are the components of the strain tensor in the $\{\boldsymbol{e}_i\}$ basis? Express your answer to only 2 significant digits.
2. Suppose one changes basis such that

$$\begin{aligned} \boldsymbol{a}_1 &= \tfrac{1}{\sqrt{3}}\boldsymbol{e}_1 + \tfrac{1}{\sqrt{3}}\boldsymbol{e}_2 + \tfrac{1}{\sqrt{3}}\boldsymbol{e}_3,, \\ \boldsymbol{a}_2 &= 0\boldsymbol{e}_1 + \tfrac{1}{\sqrt{2}}\boldsymbol{e}_2 - \tfrac{1}{\sqrt{2}}\boldsymbol{e}_3,, \\ \boldsymbol{a}_3 &= -\tfrac{2}{\sqrt{6}}\boldsymbol{e}_1 + \tfrac{1}{\sqrt{6}}\boldsymbol{e}_2 + \tfrac{1}{\sqrt{6}}\boldsymbol{e}_3, \end{aligned}$$

   where $\boldsymbol{a}_i$ represent new basis vectors. What are the components of of the strain tensor in the $\boldsymbol{a}_i$ basis?

50. **Spectral Representation and Transformation** You are told that the principal strains at a point are $(0.0000, 0.9875, 81.0125) \times 10^{-6}$ and that the corresponding eigenvectors in the $\{\boldsymbol{e}_i\}$ basis are

$$\boldsymbol{n}^{(1)} \rightarrow \begin{pmatrix} 0.0000 \\ 0.8944 \\ -0.4472 \end{pmatrix}, \quad \boldsymbol{n}^{(2)} \rightarrow \begin{pmatrix} 0.9937 \\ -0.0503 \\ -0.1006 \end{pmatrix}, \quad \boldsymbol{n}^{(3)} \rightarrow \begin{pmatrix} 0.1125 \\ 0.4444 \\ 0.8888 \end{pmatrix}.$$

1. What are the components of the strain tensor in the $\{\boldsymbol{e}_i\}$ basis? Express your answer to only 2 significant digits.

2. Suppose one changes basis such that

$$\begin{aligned}
\boldsymbol{a}_1 &= \tfrac{1}{\sqrt{3}}\boldsymbol{e}_1 + \tfrac{1}{\sqrt{3}}\boldsymbol{e}_2 + \tfrac{1}{\sqrt{3}}\boldsymbol{e}_3,, \\
\boldsymbol{a}_2 &= 0\boldsymbol{e}_1 + \tfrac{1}{\sqrt{2}}\boldsymbol{e}_2 - \tfrac{1}{\sqrt{2}}\boldsymbol{e}_3,, \\
\boldsymbol{a}_3 &= -\tfrac{2}{\sqrt{6}}\boldsymbol{e}_1 + \tfrac{1}{\sqrt{6}}\boldsymbol{e}_2 + \tfrac{1}{\sqrt{6}}\boldsymbol{e}_3\,,
\end{aligned}$$

where $\boldsymbol{a}_i$ represent new basis vectors. What are the components of the strain tensor in the $\boldsymbol{a}_i$ basis?

51. **Spectral Representation and Transformation** You are told that the principal strains at a point are $(-6.4752, 0.2270, 12.2483) \times 10^{-6}$ and that the corresponding eigenvectors in the $\{\boldsymbol{e}_i\}$ basis are

$$\boldsymbol{n}^{(1)} \to \begin{pmatrix} -0.6553 \\ 0.3874 \\ 0.6485 \end{pmatrix}, \quad \boldsymbol{n}^{(2)} \to \begin{pmatrix} 0.1319 \\ -0.7866 \\ 0.6032 \end{pmatrix}, \quad \boldsymbol{n}^{(3)} \to \begin{pmatrix} 0.7438 \\ 0.4808 \\ 0.4643 \end{pmatrix}.$$

1. What are the components of the strain tensor in the $\{\boldsymbol{e}_i\}$ basis? Express your answer to only 2 significant digits.
2. Suppose one changes basis such that

$$\begin{aligned}
\boldsymbol{a}_1 &= \tfrac{\sqrt{3}}{2}\boldsymbol{e}_1 + \tfrac{1}{4}\boldsymbol{e}_2 + \tfrac{\sqrt{3}}{4}\boldsymbol{e}_3,, \\
\boldsymbol{a}_2 &= -\tfrac{1}{2}\boldsymbol{e}_1 + \tfrac{\sqrt{3}}{4}\boldsymbol{e}_2 + \tfrac{3}{4}\boldsymbol{e}_3,, \\
\boldsymbol{a}_3 &= 0\boldsymbol{e}_1 - \tfrac{\sqrt{3}}{2}\boldsymbol{e}_2 + \tfrac{1}{2}\boldsymbol{e}_3\,,
\end{aligned}$$

where $\boldsymbol{a}_i$ represent new basis vectors. What are the components of the strain tensor in the $\boldsymbol{a}_i$ basis?
3. Verify that the three invariants of the strain tensor are the same in both bases.

52. **Extremal Normal Strains** You are told that the (small) strain tensor at a particular point in a body is given by

$$\begin{aligned}
\boldsymbol{\varepsilon} = [&\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + 2(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1) - 6(\boldsymbol{e}_1 \otimes \boldsymbol{e}_3 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_1) \\
&3\boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + 2(\boldsymbol{e}_2 \otimes \boldsymbol{e}_3 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_2) - 3\boldsymbol{e}_3 \otimes \boldsymbol{e}_3] \times 10^{-4}\,.
\end{aligned}$$

Determine the algebraically maximum and minimum normals strains at this point and their corresponding directions.

53. **Volume Average Strains** Define the volume average strain in a body by

$$\bar{\boldsymbol{\varepsilon}} = \frac{1}{V}\int_{\Omega} \boldsymbol{\varepsilon}\, d\Omega$$

where V is the volume of the body Ω. Show that

$$\bar{\boldsymbol{\varepsilon}} = \frac{1}{2V}\int_{\partial\Omega} \boldsymbol{u} \otimes \boldsymbol{n} + \boldsymbol{n} \otimes \boldsymbol{u}\, d\Gamma$$

54. **Volume Average Strains** Consider the 2-D plane strain disk, Ω, of radius R as shown. The disk contains an elastic-plastic (regular) hexagonal inclusion of side length a. The moduli of the inclusion are μ_i and λ_i. The yield stress is σ_Y and there is no hardening. The yield function is given by Mises relation and the flow rule is associative. The inclusion is surrounded by an elastic material with moduli μ_m and λ_m. A load has been applied to the disk and the displacements on the outer boundary, $\partial\Omega$, have been measured as

$$\boldsymbol{u}(R,\theta) = A\sin^2(\theta)\boldsymbol{e}_r + B\cos^2(\theta)\boldsymbol{e}_\theta\,. \tag{59}$$

Find the average strain in the disk; i.e., determine

$$\bar{\boldsymbol{\varepsilon}} = \frac{1}{\pi R^2}\int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{x})\,d\boldsymbol{x} \tag{60}$$