Questions on Vectors and Tensors

1. **Basic Operations** Consider two vectors $\boldsymbol{a} \sim (1, 4, 6)$ and $\boldsymbol{b} \sim (2, 0, 4)$, where the components have been expressed in a given orthonormal basis. Compute

    1. $\|\boldsymbol{a}\|$.
    2. The angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.
    3. The area of the parallelogram bounded by $\boldsymbol{a}$ and $\boldsymbol{b}$.
    4. $\boldsymbol{b} \times \boldsymbol{a}$.

2. **Projection** Denote the basis in Problem 1 as $\{\boldsymbol{e}_i\}_{i=1}^{3}$. What is the component of $\boldsymbol{a}$ in the direction $\boldsymbol{z} = \boldsymbol{e}_1 + \boldsymbol{e}_2$.

3. **Basic Operations** As we have seen we can express vectors in terms of their components with respect to a particular basis. Thus we can write

$$\boldsymbol{v} = v_i \boldsymbol{e}_i \,.$$

When the particular basis is understood (ie. everyone reading the equation will assume the same basis) one can safely express the components of the vector in a "column vector"; ie. as

$$v_i \rightarrow \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} .$$

This convention also applies to second order tensors. In this case we have for a basis $e = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$

$$\boldsymbol{T} = T_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \,.$$

If the basis is understood we can express the components in a 3×3 matrix. The following ordering is conventional:

$$T_{ij} \rightarrow \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

For the problems assume an orthonormal basis $e = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ and

$$\begin{aligned} \boldsymbol{a} &= 2\boldsymbol{e}_1 + 5\boldsymbol{e}_2 - 7\boldsymbol{e}_3 \,, \\ \boldsymbol{b} &= 0\boldsymbol{e}_1 - 8\boldsymbol{e}_2 + 1\boldsymbol{e}_3 \,, \end{aligned}$$

$$c_i \rightarrow \begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix} ,$$

and

$$T_{ij} \rightarrow \begin{bmatrix} 1 & 8 & 2 \\ 8 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}.$$

Find expressions for the following. If the operation is a scalar just report the number. If the operation results in a vector, give it in dyadic form (component values times basis vectors, summed over all needed basis vectors) as well as in column vector form assuming the e basis. If the operation results in a tensor, give it in dyadic form (component values times $\boldsymbol{e}_1 \otimes \boldsymbol{e}_1$ etc. summed over all needed $\boldsymbol{e}_i \otimes \boldsymbol{e}_j$) as well as in matrix form assuming the e basis.

(a) $(\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$, (b) $b_3\boldsymbol{a} \cdot \boldsymbol{c}$, (c) $(\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{a} \otimes \boldsymbol{b}$, (d) δ_{ii}, (e) $T_{3j}\delta_{3j}$, (f) $T_{ij}\delta_{ij}$, (g) $T_{ij}T_{ij}$, (h) $\boldsymbol{Tc}$, (i) I_T, (j) II_T, (k) III_T, (l) $\boldsymbol{e}_1 \otimes \boldsymbol{e}_2$, (m) $\boldsymbol{e}_3 \otimes \boldsymbol{e}_2$.

4. **Tensor Operations** Let $\boldsymbol{a} = 1\boldsymbol{e}_1 + 4\boldsymbol{e}_2 + 6\boldsymbol{e}_3$, $\boldsymbol{b} = 7\boldsymbol{e}_1 + 2\boldsymbol{e}_2 + 3\boldsymbol{e}_3$, $\boldsymbol{c} = 1\boldsymbol{e}_2 + 2\boldsymbol{e}_3$, and $\boldsymbol{T} = 12\boldsymbol{a} \otimes \boldsymbol{b} + 7\boldsymbol{a} \otimes \boldsymbol{c}$. Determine the following

   1. $\boldsymbol{T} \cdot \boldsymbol{T}$
   2. $\boldsymbol{T}\boldsymbol{e}_1$
   3. $\boldsymbol{T}^T$

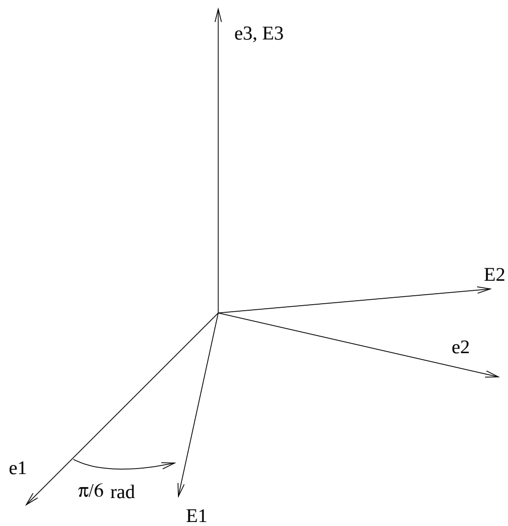
5. **Tensor-Vector Operations** *[This is a two dimensional problem.]* Let the components of a 2nd order tensor $\boldsymbol{A}$ be $A_{11} = 1$, $A_{12} = A_{21} = 2$, $A_{22} = 3$. Let the components of a vector $\boldsymbol{v}$ be $v_1 = 1$, $v_2 = 2$.

   1. Compute the components of $\boldsymbol{w} = \boldsymbol{A} \cdot \boldsymbol{v}$
      (a) using the relation $w_i = A_{ij}v_j$,
      (b) using matrix vector multiplication rules.
   2. Compute the scalar $s = \boldsymbol{v} \cdot \boldsymbol{A} \cdot \boldsymbol{v} \equiv \boldsymbol{A}(\boldsymbol{v}, \boldsymbol{v})$
      (a) using the relation $s = A_{ij}v_iv_j$,
      (b) using matrix vector multiplication.

6. **Orthogonality of Skew and Symmetric Tensors** If $\boldsymbol{A}$ is a symmetric tensor and $\boldsymbol{B}$ is skew-symmetric tensor, show $\mathrm{tr}[\boldsymbol{A}^T\boldsymbol{B}] = A_{ij}B_{ij} = 0$.

7. **Symmetric Contraction** Consider a tensor $\boldsymbol{A} \in \mathbb{S}^3$ (i.e. symmetric) and an arbitrary tensor $\boldsymbol{B}$. Prove that $\boldsymbol{A} : \boldsymbol{B} = \boldsymbol{A} : \mathrm{sym}[\boldsymbol{B}]$.

8. **Vector Component Extraction** Given $\boldsymbol{a} = 3\boldsymbol{e}_1 + 4\boldsymbol{e}_2 + 5\boldsymbol{e}_3$ find a_A in the $\boldsymbol{E}_A$ basis as defined in the following figure. Do this by using our component extraction rules.



9. **Tensor Component Extraction** Consider a 2nd order tensor $\boldsymbol{T}$ whose components in the $\boldsymbol{e}_i$ basis are given by

$$\boldsymbol{T} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 5 \\ 2 & 4 & 0 \end{bmatrix}$$

Using the component extraction rule for tensors, find T_{12} and T_{33} in the $\boldsymbol{E}_A$ basis. Assume $\boldsymbol{e}_i$ and $\boldsymbol{E}_A$ are as defined in Problem 8. [Do not try and assemble the rotation tensor. It is a waste of time for this problem.]

10. **Basis conversion for tensor components** Consider the tensor $\boldsymbol{T} = T_{ij}\hat{\boldsymbol{e}}_i \otimes \hat{\boldsymbol{e}}_j$, where

$$\boldsymbol{T} \sim \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{\{\hat{\boldsymbol{e}}_i\}}$$

Consider a second basis $\{\hat{\boldsymbol{E}}_A\}$ whose relation to the basis $\{\hat{\boldsymbol{e}}_i\}$ is given by

$$\hat{\boldsymbol{E}}_1 = \frac{1}{2}\hat{\boldsymbol{e}}_1 + \frac{\sqrt{3}}{2}\hat{\boldsymbol{e}}_2$$
$$\hat{\boldsymbol{E}}_2 = -\frac{\sqrt{3}}{2}\hat{\boldsymbol{e}}_1 + \frac{1}{2}\hat{\boldsymbol{e}}_2\,.$$

Find the value of T_{11} in the $\{\hat{\boldsymbol{E}}_A\}$ basis.

11. **Identity Representation** Consider an arbitrary vector Starting from $\boldsymbol{a} = \boldsymbol{a}$ show that the identity tensor can be expressed as $\mathbf{1} = \boldsymbol{e}_i \otimes \boldsymbol{e}_i$ where $\{\boldsymbol{e}_i\}_{i=1}^3$ represents any orthonormal basis. Not that the identity tensor is defined to be the tensor $\mathbf{1}$ such that $\boldsymbol{a} = \mathbf{1}\boldsymbol{a}$ for all vectors $\boldsymbol{a}$.

12. **Cross Product** Let $\boldsymbol{v} = 1\boldsymbol{e}_1 + 2\boldsymbol{e}_2$, $\boldsymbol{w} = 2\boldsymbol{e}_3 + 2\boldsymbol{e}_1 + 1\boldsymbol{e}_2$ and $\boldsymbol{z} = \boldsymbol{v} \times \boldsymbol{w}$. What is z_3? Do this problem using indicial notation methods. Do NOT use the classical technique with the determinant.

13. **Permutation Symbol** Show that $e_{ijk}e_{ijk} = 6$.

14. **Orthogonal Tensor** In two dimensions, any orthogonal tensor can be expressed as

$$\begin{aligned}\boldsymbol{R} &= \cos(\theta)\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \sin(\theta)\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 \\ &\quad - \sin(\theta)\boldsymbol{e}_2 \otimes \boldsymbol{e}_1 + \cos(\theta)\boldsymbol{e}_2 \otimes \boldsymbol{e}_2 \,.\end{aligned}$$

   1. Show that $\boldsymbol{R}^T\boldsymbol{R} = \mathbf{1}$; ie. prove that the inverse of an orthogonal tensor is its transpose.
   2. Show that $\|\boldsymbol{v}\| = \|\boldsymbol{R}\boldsymbol{v}\|$ for all $\boldsymbol{v}$; ie. an orthogonal tensor does not change the length of a vector.

15. **Rodriguez Formula** Let $\boldsymbol{\Lambda} \in SO(3)$ (i.e. a rotation tensor).

   (a)Argue that $\boldsymbol{\Lambda}$ admits the following representation:

$$\boldsymbol{\Lambda} = \boldsymbol{e}_3 \otimes \boldsymbol{e}_3 + \sin(\theta)(\boldsymbol{e}_2 \otimes \boldsymbol{e}_1 - \boldsymbol{e}_1 \otimes \boldsymbol{e}_2) + \cos(\theta)(\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_2) \qquad (*)$$

   where $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ form an orthonormal basis. [Hint: Every rotation has a vector $\boldsymbol{e}_3$ such that $\boldsymbol{\Lambda}\boldsymbol{e}_3 = \boldsymbol{e}_3$.]

   (b) Using the representation in part (a) show that one also has the following representation for $\boldsymbol{\Lambda}$:

$$\boldsymbol{\Lambda} = \mathbb{I} + \frac{\sin(\theta)}{\theta}\boldsymbol{\Theta} + \frac{1}{2}\frac{\sin^2(\theta/2)}{(\theta/2)^2}\boldsymbol{\Theta}^2 \qquad (\#)$$

   where $\boldsymbol{\Theta}$ is the skew-symmetric tensor with axial vector $\check{\boldsymbol{\Theta}} = \theta\boldsymbol{e}_3$. [Remark: Formula (#) is often referred to as Rodriguez' formula. This representation plays a crucial role in non-linear computational mechanics.]

16. **Indices** How many indices will a rank 3 tensor have?

17. Tensor Classification Classify $A_{ij}B_jC_iD_{lmn}X_{mn}Y_l$ as the components of a scalar, vector, or higher order tensor.

18. **Tensor Classification** Classify $A_ib_{ikl}C_{mnpp}d_{ln}$ as the components of a scalar, vector, or higher order tensor.

19. **3rd Order Tensor** Let φ be a third order tensor that is skew-symmetric; i.e.

$$\varphi(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = -\varphi(\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{w}) = -\varphi(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}) = -\varphi(\boldsymbol{w}, \boldsymbol{v}, \boldsymbol{u})$$

   for all vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$. Let $\boldsymbol{S}$ be a second order tensor; show that

$$\varphi(\boldsymbol{S}\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3) + \varphi(\boldsymbol{e}_1, \boldsymbol{S}\boldsymbol{e}_2, \boldsymbol{e}_3) + \varphi(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{S}\boldsymbol{e}_3) = \mathrm{tr}[\boldsymbol{S}]\varphi(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$$

   [Hint: First show $\varphi(\boldsymbol{S}\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3) = S_{11}\varphi(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$.]

20. **Components of the isotropic elasticity tensor** Consider the 4th order tensor $\mathbb{C}$ with components $\lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$, where λ and μ are given constants.

    1. Determine the value of $\mathbb{C}_{1122} = \mathbb{C}(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_2)$.
    2. Determine the value of $\mathbb{C}_{1112} = \mathbb{C}(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2)$.

21. **Basic Field Operations** Let $\boldsymbol{v}(\boldsymbol{x})$ and $\boldsymbol{w}(\boldsymbol{x})$ be two vector fields and $\boldsymbol{S}(\boldsymbol{x})$ be a tensor field. Convert the following expressions to indicial or symbolic notation as appropriate.

    1. $\text{div}[\boldsymbol{S}] \cdot \boldsymbol{w}$
    2. $\boldsymbol{v} \otimes (\boldsymbol{S}\boldsymbol{w})$
    3. $v_i w_j w_{k,k}$
    4. $w_j S_{ij} u_{l,i}$

22. **Eigenvalues** Consider the scalar $\boldsymbol{x} \cdot \boldsymbol{S}\boldsymbol{x}$ for symmetric $\boldsymbol{S}$. Show that the necessary equations that this quantity be a maximum over all unit vectors $\boldsymbol{x}$ is given by the eigenvalue equation $\boldsymbol{S}\boldsymbol{x} = s\boldsymbol{x}$. To do this extremize $\boldsymbol{x} \cdot \boldsymbol{S}\boldsymbol{x}$ with respect to $\boldsymbol{x}$ subject to the constraint $\|\boldsymbol{x}\| = 1$ using a Lagrange multiplier method.

23. **Invariants** Consider the tensor $\boldsymbol{S} = 2(\boldsymbol{e}_2 \otimes \boldsymbol{e}_1 + \boldsymbol{e}_1 \otimes \boldsymbol{e}_2)$. Compute the principal invariants of $\boldsymbol{S}$.

24. **Spectral Form** Consider the tensor $\boldsymbol{S} = 1(\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_2) + 2(\boldsymbol{e}_2 \otimes \boldsymbol{e}_1 + \boldsymbol{e}_1 \otimes \boldsymbol{e}_2)$. Express $\boldsymbol{S}$ in spectral form.

25. **Directional Derivative** Given a stress field $\boldsymbol{\sigma}(\boldsymbol{x})$ on a domain $\Omega \subset \mathbb{R}^3$, explain in words the meaning of $\mathrm{D}_{\boldsymbol{v}}\boldsymbol{\sigma}(\boldsymbol{x}) = \frac{d}{d\alpha}\Big|_{\alpha=0} \boldsymbol{\sigma}(\boldsymbol{x} + \alpha\boldsymbol{v})$, where $\boldsymbol{v} \in \mathbb{R}^3$ is given. In particular, your answer should allow one to understand the meaning of the components of $\mathrm{D}_{\boldsymbol{v}}\boldsymbol{\sigma}(\boldsymbol{x})$.

26. **Directional Derivative** Compute $D\boldsymbol{G}(\boldsymbol{A})$ where $\boldsymbol{G}(\boldsymbol{A}) = \text{tr}(\boldsymbol{A})\boldsymbol{A}$.

27. **Directional Derivative of 2nd Invariant** The second invariant of a rank 2 tensor $\boldsymbol{A}$ is given by $I_2(\boldsymbol{A}) = \frac{1}{2}[A_{ii}A_{jj} - A_{kl}A_{lk}]$. Determine the directional derivative of $I_2(\boldsymbol{A})$ in the direction $\boldsymbol{H}$. Infer from this result the expression for $\frac{\partial}{\partial A_{ij}} I_2(\boldsymbol{A})$ by applying the relation
$$\frac{\partial I_2(\boldsymbol{A})}{\partial A_{ij}} H_{ij} = DI_2(\boldsymbol{A})[\boldsymbol{H}]$$

28. **Directional Derivative** Consider $f(\boldsymbol{x}) = x_1x_2x_3 + x_1x_2$

    1. Compute the directional derivative of f in the direction $\hat{\boldsymbol{e}}_1 + \hat{\boldsymbol{e}}_2 + \hat{\boldsymbol{e}}_3$ at the point $(1, 2, 3)$. [Warning! Normalize!]
    2. Compute the gradient of f – i.e. ∇f.

29. **Directional derivative** Consider $f(\boldsymbol{x}) = x_1x_2x_3$

1. Compute the directional derivative of f in the direction of $\hat{\boldsymbol{e}}_1 + \hat{\boldsymbol{e}}_2$ at the point $(2, 2, 2)$. [Warning! Normalize!].
2. Compute the gradient of f.

30. **Gradient of a vector field** Consider a vector field $\boldsymbol{v} = v_i \hat{\boldsymbol{e}}_i$, where $v_1 = x_1^2$, $v_2 = x_2$, and $v_3 = x_3 x_1$. Compute the components of $\nabla \boldsymbol{v}$.

31. **Directional Derivative** Consider $\boldsymbol{g}(\boldsymbol{x}) = 4(x_1)^2 \hat{\boldsymbol{e}}_1 + x_1 x_2 x_3 \hat{\boldsymbol{e}}_2 + x_3 \hat{\boldsymbol{e}}_3$ and compute $D\boldsymbol{g}[\hat{\boldsymbol{h}}]$ at the point $(0, 0, 0)$ where $\hat{\boldsymbol{h}} = (\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3)/\sqrt{3}$

32. **Directional Derivative** The definition of the derivative of a scalar valued function of a second order tensor is the operator $\frac{\partial f}{\partial \boldsymbol{A}}$ such that

$$\left.\frac{d}{d\alpha}\right|_{\alpha=0} f(\boldsymbol{A} + \alpha \boldsymbol{H}) = \frac{\partial f}{\partial A_{ij}} H_{ij} \,.$$

Use this definition to show that

1. $\frac{\partial \frac{1}{3}\mathrm{tr}[\boldsymbol{\sigma}]}{\partial \boldsymbol{\sigma}} = \frac{1}{3}\mathbf{1}$
2. $\frac{\partial(\varepsilon_{ij}\varepsilon_{ij})}{\partial \varepsilon_{kl}} = 2\varepsilon_{kl}$

33. **Gradient** Consider $f : \mathbb{S}^3 \to \mathbb{R}$ of the form $f(\boldsymbol{\sigma}) = \|\boldsymbol{s}\| - (R + k\,\mathrm{tr}[\boldsymbol{\sigma}])$, where $\boldsymbol{s}$ is the deviatoric part of $\boldsymbol{\sigma}$ and R and k are constants. Find $\frac{\partial f}{\partial \boldsymbol{\sigma}}$.

34. **Gradient** Consider $f : \mathbb{R}^3 \to \mathbb{R}$ where $f(\boldsymbol{x}) = x_k x_i A_{ki} + e_{ijk} x_j c_k d_i$. $\boldsymbol{A}$, $\boldsymbol{c}$, and $\boldsymbol{d}$ are given constant tensors and vectors.

1. Compute the components of the gradient of $f(\cdot)$; i.e., find $\partial f / \partial x_i$.
2. Express $\partial f / \partial \boldsymbol{x}$ in symbolic form.

35. **Directional Derivative** Determine the rate of change of the following tensor valued function:

$$\boldsymbol{T}(\boldsymbol{x}) = [5\boldsymbol{x} \cdot \boldsymbol{x} + \frac{1}{2}\boldsymbol{x} \cdot \boldsymbol{A}\boldsymbol{x}]\boldsymbol{x} \otimes \boldsymbol{x}$$

at the point $\boldsymbol{x} = \hat{\boldsymbol{e}}_1 + \hat{\boldsymbol{e}}_3$ in the direction $\hat{\boldsymbol{h}} = (\hat{\boldsymbol{e}}_1 + \hat{\boldsymbol{e}}_2 + \hat{\boldsymbol{e}}_3)/\sqrt{3}$, where $\boldsymbol{A} = \hat{\boldsymbol{e}}_1 \otimes \hat{\boldsymbol{e}}_1 + \hat{\boldsymbol{e}}_2 \otimes \hat{\boldsymbol{e}}_3 + \hat{\boldsymbol{e}}_3 \otimes \hat{\boldsymbol{e}}_2$.

36. **Gradient** Consider $f : \mathbb{E}^3 \to \mathbb{R}$ where $f(\boldsymbol{x}) = x_1^2 + 2x_1 x_2^2 + 3x_2^2 x_3$. Find $\mathrm{grad}(f)$.

37. **Gradient** Consider $f(\boldsymbol{x}) = x_1^2 + x_1 x_2 x_3$. Compute ∇f

38. **Directional Derivative** Consider $f(\boldsymbol{x}) = 4x_1 + x_1 x_2 x_3 + \cos(x_2)$.

1. Compute the directional derivative of f in the direction $\boldsymbol{e}_1 + \boldsymbol{e}_2$ at the point (1,1,1).
2. Compute the gradient of f, ∇f.

39. **Directional Derivative** Consider $f(\boldsymbol{x}) = 4(x_1)^2 + x_1 x_2 x_3$.

    1. Compute the directional derivative of f in the direction $\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3$ at the point (0,1,1). [Warning! Normalize!]
    2. Compute the gradient of f, ∇f.

40. **Divergence theorem** Use the Divergence Theorem to show that

$$\int_{\partial \mathcal{R}} \frac{1}{3} \boldsymbol{x} \cdot \hat{\boldsymbol{n}} \, dA = \mathrm{vol}[\mathcal{R}] \,,$$

where $\mathcal{R}$ is a simply connected region of space with boundary $\partial \mathcal{R}$ and outward unit normal $\hat{\boldsymbol{n}}$.

41. **Divergence Theorem** Use the Divergence Theorem to show that

$$\int_{\mathcal{R}} 5 x_i x_j \, dV = \int_{\partial \mathcal{R}} x_i x_j x_k \hat{n}_k \, dA \,.$$

42. **Divergence Theorem** Consider a ball, B, of radius 4 centered at the origin. Using the divergence theorem, compute

$$\int_{\partial B} \boldsymbol{x} \cdot \boldsymbol{n} \, dA \,,$$

where $\boldsymbol{x}$ is (as usual) the position vector and $\boldsymbol{n}$ is the unit outward normal to the surface ∂B.

43. **Divergence theorem** Compute the value of the integral

$$\int_{\partial \mathcal{R}} \hat{\boldsymbol{n}} \, dA \,,$$

where $\mathcal{R}$ is a simply connected region of space with boundary $\partial \mathcal{R}$ and outward unit normal $\hat{\boldsymbol{n}}$.

44. **Divergence Theorem** Consider the unit ball $\mathcal{B} = \{\boldsymbol{x} \mid \|\boldsymbol{x}\| < 1\}$ with boundary $\partial \mathcal{B} = S^2 = \{\boldsymbol{x} \mid \|\boldsymbol{x}\| = 1\}$. Using the divergence theorem compute

$$\int_{\mathcal{B}} \mathrm{div} \left(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} \right) \, dV \,.$$