Mechanics of Structures (CE130N)
Lab 7

1 Objective

The objective of this lab is to program a Ritz solution using a special set of Ritz functions known as hat functions. These function underly the most important numerical method for solving a very wide variety of problems in science and engineering – viz., the *finite element method*.

2 Model Problem

The model problem we will work with is an elastic tension-compression bar which has an imposed displacement at the left-end, an applied force at the right-end, and a constant distributed load. Thus the problem we wish to solve looks like:

$$AEu'' + b = 0\,, \tag{1}$$

where $u(0) = 0.03$, $AE\frac{du}{dx}(l) = F_{\text{app}} = -2000$, $b(x) = 4000$ and $AE = 30 \times 10^6$ – all in US customary units. The total potential energy which describes this problem is

$$\Pi(u(x)) = \int_0^l \frac{1}{2}AE(u')^2\,dx - \int_0^l bu\,dx - F_{\text{app}}u(l)\,, \tag{2}$$

where $l = 2$.

3 Linear Hat Functions

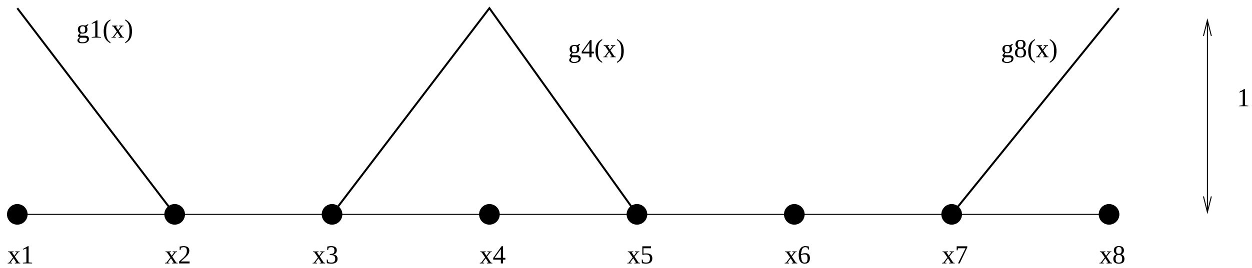
As an approximation we will assume that

$$u(x) = \sum_{I=1}^{n} c_I g_I(x)\,, \tag{3}$$

where

$$g_I(x) = \begin{cases} \frac{x - x_{I-1}}{\Delta x} & x_{I-1} < x < x_I \\ \frac{x_{I+1} - x}{\Delta x} & x_I < x < x_{I+1} \\ 0 & \text{otherwise}\,. \end{cases} \tag{4}$$

The points $x_I = (I-1)\Delta x$ are called the nodes and $\Delta x = l/(n-1)$, where n is the number of functions in our Ritz expansion. Note that each function is equal to one at the node associated with its index. The intervals between the nodes are called the elements and there are $nel = n-1$ of them. Graphically these functions look as follows:



In the figure, we have used the example of $n = 8$ and thus $\Delta x = 2/7$ and $nel = 7$. Note that the functions on the ends are non-zero over just one element whereas those in the interior are non-zero over two elements.

4 Discrete Equations

Inserting the Ritz expansion into the functional and taking its derivative with respect to an arbitrary parameter yields the system of equations:

$$\sum_{J=1}^{n} K_{IJ} c_J = F_I \, , \tag{5}$$

where

$$K_{IJ} = \int_0^l g_I' A E g_J' \, dx \tag{6}$$

and

$$F_I = \int_0^l g_I b \, dx + F_{\text{app}} g_I(2) \tag{7}$$

Because the linear hat functions have compact support most of the terms in the stiffness matrix are zero. In fact $K_{IJ} = 0$ unless $|I - J| \leq 1$. It should also be observed that we already know that $c_1 = 0.03$ by the boundary conditions.

To efficiently compute the integrals and implement them in a clean fashion in code, the integrals are computed by computing the contributions from each element and then assembling them into the global stiffness K_{IJ}. Thus

$$K_{IJ} = \int_0^l g_I' A E g_J' \, dx = \sum_{L=1}^{nel} \int_{x_L}^{x_{L+1}} g_I' A E g_J' \, dx \, , \tag{8}$$

where nel is the total number of elements. The term inside the summation is known as the element stiffness and in our case it has only four non-zero entries. These occur for

$I, J \in \{L, L+1\}$. Thus when computing the contribution from a given element, say element L, we usually form a two-by-two matrix with entries:

$$k_{ij}^e = \int_{x_L}^{x_{L+1}} g_i^{e'} AE g_j^{e'} \, dx \,, \tag{9}$$

where $i, j \in \{1, 2\}$ and $g_1^e = g_L$ and $g_2^e = g_{L+1}$. In this way k_{ij}^e is a two-by-two matrix that contributes to the global stiffness matrix. For example, consider a case where one has 4 elements. Then for element two $k_{11}^{e=2}$ would contribute to as $K_{22} = K_{22} + k_{11}^{e=2}$, $K_{23} = K_{23} + k_{12}^{e=2}$, etc.

We treat the right hand side in a similar fashion; i.e. we form an element right-hand side and assemble it into the global right-hand side. So considering element L, we have

$$f_i^e = \int_{x_L}^{x_{L+1}} g_i^e b \, dx \, + F_{\text{app}} g_i^e(2) \,. \tag{10}$$

Here as before $i \in \{1, 2\}$ and $g_1^e = g_L$ and $g_2^e = g_{L+1}$. This two-by-one vector is then assembled into the global right-hand side. For example, consider a case where one has 4 elements. Then for element three $F_3 = F_3 + f_1^{e=3}$ and $F_4 = F_4 + f_2^{e=3}$.

4.1 Exercise 1

Compute, by hand, an expression for the matrix elements k_{ij}^e for a generic element. Your result should be a two-by-two matrix and it will be symmetric.

4.2 Exercise 2

Compute an expression for f_i^e for a generic element. Your result should be a two-by-one vector. Note that the applied end force will only affect the very last element.

4.3 Exercise 3

Download the file `lab7_student.m` from bspace and program your expressions into it to solve the given problem. If you have done it correctly you will find that your approximate solution is *exact* at the nodes.

1. Test it for $n = 2$ and $n = 3$ to verify this.

2. How many elements do you for the error to vanish in the "eye" norm?

4.4 Exercise 4: Extra if you have time

Make a log-log plot of the relative L^2 error versus n. How many terms are required to reduce the relative error to 10^{-6}? Hint: Compute the integrals by performing numerical quadrature over the elements and then add up the result.