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Department of Civil Engineering
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Structural Engineering,
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Mechanics of Structures (CE130N)
Lab 2

1 Objective

The objective of this lab is to apply the skills you learned in Lab 1 to the case of beam bending. You will first modify your program so that it is able to solve beam bending problems and plot appropriate results. Then you will apply the program to two simple problems for which hand solutions are well-known. The program from this lab will also help you with up-coming homework assignments.

2 Beam equations in first order form

2.1 Governing differential equations

The governing equations for a beam in bending are:

- Equilibrium

$$\frac{dV}{dx} + q(x) = 0, \tag{1}$$

$$\frac{dM}{dx} + V = 0\,, \tag{2}$$

where $q(x)$ is the distributed load.

- Kinematics

$$\theta = \frac{dv}{dx}, \tag{3}$$

$$\kappa = \frac{d\theta}{dx}\,. \tag{4}$$

- Effective constitutive relation

$$M \quad = \quad E(x)I(x)\kappa \tag{5}$$

The Young's modulus $E(x)$ and second moment of inertia $I(x)$ can vary along the length of the beam.

These relations can be combined to obtain a single equation representing equilibrium in terms of the displacement,

$$\frac{d^2}{dx^2}\left[EI\frac{d^2v}{dx^2}\right] = q\,. \tag{6}$$

This relation is effective for hand solutions as we have seen lecture and in the homework. In order to utilize the solver in MATLAB, one needs to employ the governing equations in first-order form:

$$\boxed{\frac{d\boldsymbol{y}}{dx} = \boldsymbol{f}(\boldsymbol{y}, x),} \tag{7}$$

where $\boldsymbol{y}$ is the vector of unknown variables, and $\boldsymbol{f}$ is a vector of known functions depending on $\boldsymbol{y}$ and the position x.

For beams in bending, we choose the variables v, θ, M, and V as the unknown variables (since the boundary conditions are typically enforced on these quantities). Recall from the solution of systems of equations that one must have the same number of equations as variables. Thus we must obtain obtain 4 differential equations from the governing equations above. These can be obtained by eliminating the variable κ from Eqns. (1-5). This yields the four equations:

$$\begin{aligned}
\frac{dV}{dx} + q(x) &= 0, \\
\frac{dM}{dx} + V &= 0, \\
\theta &= \frac{dv}{dx}, \\
M &= E(x)I(x)\frac{d\theta}{dx}\,.
\end{aligned}$$

These can be rewritten as:

$$\frac{d}{dx}\begin{bmatrix} v \\ \theta \\ M \\ V \end{bmatrix} = \begin{bmatrix} \theta \\ \dfrac{M}{E(x)I(x)} \\ -V \\ -q(x) \end{bmatrix}.$$

By defining

$$\begin{aligned}
\boldsymbol{y} &:= \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} v \\ \theta \\ M \\ V \end{bmatrix}, \\
\boldsymbol{f}(\boldsymbol{y}, x) &:= \begin{bmatrix} f_1(\boldsymbol{y}, x) \\ f_2(\boldsymbol{y}, x) \\ f_3(\boldsymbol{y}, x) \\ f_4(\boldsymbol{y}, x) \end{bmatrix} = \begin{bmatrix} y_2 \\ \dfrac{y_3}{E(x)I(x)} \\ -y_4 \\ -q(x) \end{bmatrix},
\end{aligned} \tag{8}$$

one obtains the desired first-order form,

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} y_2 \\ \dfrac{y_3}{E(x)I(x)} \\ -y_4 \\ -q(x) \end{bmatrix}.$$

2.2 Boundary condition

To solve the system of differential equations, one must apply boundary conditions. For the fourth-order differential equation (6), one requires 4 boundary conditions.

In order to apply boundary conditions in the solver in MATLAB, one needs to define a function which returns a residual measuring the the boundary condition violation; a residual of zero implies that the boundary conditions are satisfied exactly. The function has the form,

$$\boxed{\boldsymbol{g}(\boldsymbol{y}(a), \boldsymbol{y}(b))}$$

where $\boldsymbol{g}$ is vector of functions depending on the value of $\boldsymbol{y}$ evaluated at the boundaries $x = a$ and $x = b$.

To clarify the form of the function, consider a cantilever beam with an end shear; assume the end at $x = 0$ is built-in and the end at $x = L$ has the applied end shear $\bar{V}_L$, then

$$\begin{aligned} v(0) &= 0, \\ \theta(0) &= 0, \\ M(L) &= 0, \\ V(L) &= \bar{V}_L . \end{aligned}$$

The vector defining the boundary condition residual is then given as

$$\begin{bmatrix} v(0) - 0 \\ \theta(0) - 0 \\ M(L) - 0 \\ V(L) - \bar{V}_L \end{bmatrix} .$$

Using the correspondence between v, θ, M, V and $\boldsymbol{y}$ defined in Eqn. (8), one can define $\boldsymbol{g}$ as:

$$\boldsymbol{g}(\boldsymbol{y}(0), \boldsymbol{y}(L)) := \begin{bmatrix} g_1(\boldsymbol{y}(0), \boldsymbol{y}(L)) \\ g_2(\boldsymbol{y}(0), \boldsymbol{y}(L)) \\ g_3(\boldsymbol{y}(0), \boldsymbol{y}(L)) \\ g_4(\boldsymbol{y}(0), \boldsymbol{y}(L)) \end{bmatrix} = \begin{bmatrix} y_1(0) - 0 \\ y_2(0) - 0 \\ y_3(L) - 0 \\ y_4(L) - \bar{V}_L \end{bmatrix} . \tag{9}$$

3 First Steps

Starting from the file from Lab 1:

1. Edit the function names in the file to make them refer to beams instead of bars. This is not necessary but is stylistically important – for example, bar1d → beam1d, etc.

2. Edit the function beam1d so that the number of variables will be 4.

3. Edit beam1d_ode to implement the beam bending equations.

4. Edit beam1d_bc to implement beam bending boundary conditions.

5. Edit beam1d_plot to create 2 additional sub-plots so that you have one plot for each of the four solution fields v, θ, M, V. Make sure that you have the correct plot labels.

3.1 Test your program

Use your program to solve the following problems.

1.

$$\begin{aligned} E &= 200 \text{ kN/mm}^2, \\ I &= 10^4 \text{ mm}^4, \\ L &= 1000 \text{ mm}, \\ q(x) &= q_0 = -10 \text{ N/mm}, \end{aligned}$$

with boundary conditions

$$\begin{aligned} u(0) &= 0, \\ M(0) &= 0, \\ u(L) &= 0, \\ M(L) &= 0\,. \end{aligned}$$

This is a beam with both ends pinned and a uniform distributed load of 1. Is the displacement at the center of the beam what you expect from the classical solution? This solution is,

$$u(L/2) = -\frac{5}{384}\frac{q_0 L^4}{EI}\,.$$

Use `deval( )` to properly evaluate the solution. Do all four plots correspond to your intuition?

2. Change the file so that you obtain the solution for the problem,

$$\begin{aligned} E &= 30 \times 10^6 \text{ psi}, \\ I &= 25 \text{ in}^4, \\ L &= 36 \text{ in}, \\ q(x) &= 0, \end{aligned}$$

with boundary conditions

$$\begin{aligned} u(0) &= 0, \\ \theta(0) &= 0, \\ M(L) &= 0, \\ V(L) &= -10 \text{ kip}\,. \end{aligned}$$

This is a cantilever beam with a load applied at the end. What should the shape of the displacement look like? Does the tip displacement match the expected result $-PL^3/3EI$? Use `deval( )` to properly evaluate the solution. Do all four fields match your expectations for this problem?