UNIVERSITY OF CALIFORNIA BERKELEY
Department of Civil Engineering
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Structural Engineering,
Mechanics and Materials
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Mechanics of Structures (CE130N)
Lab 6

1 Objective

The objective of this lab is to develop a understanding of the principle of stationary potential energy through the visualization of the potential energy in 1 and 2 degree of freedom systems.

2 Principle of stationary potential energy

Define the following quantites,

Π_{total}	: Total potential energy of the mechanical system,
Π_{elastic}	: Elastic energy in the mechanical system,
Π_{load}	: Energy due to the load,

and define,

$$\Pi_{\text{total}} := \Pi_{\text{elastic}} + \Pi_{\text{load}} \ .$$

Assume that all the energy quantites noted above depend on N variables or displacements, $u_1, \cdots, u_N$, i.e.,

$$\Pi_{\text{total}}(u_1, \cdots, u_N) \ .$$

By defining the vector $\boldsymbol{u}$,

$$\boldsymbol{u} := \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} \ ,$$

we can denote the dependence as,

$$\Pi_{\text{total}}(\boldsymbol{u}) \ .$$

The principle of stationary potential energy states:

$$\boxed{\begin{array}{c}\text{A mechanical system is in an equilibrium state}\\ \Leftrightarrow\\ \Pi_{\text{total}}\text{ is stationary.}\end{array}}$$

More concretely this implies the following,

$$\boxed{\begin{array}{c}\text{A mechanical system is in an equilibrium state at }\widehat{\boldsymbol{u}}\\ \Leftrightarrow\\ \dfrac{\partial\Pi_{\text{total}}}{\partial u_i}(\widehat{\boldsymbol{u}}) = 0\text{ for }i = 1,\ldots,N\end{array}}$$

This principle allows us to look for the states of equilibrium of the mechanical system by looking for the stationary points of the potential energy. This principle also tells us if that there are no stationary points, then there are no states of equilibrium.

3 Potential energy for linear mechanical systems

For linear mechanical systems under the action of dead-loads, Π_{total} can be written as,

$$\Pi_{\text{total}} = \frac{1}{2}\boldsymbol{u}^T\boldsymbol{K}\boldsymbol{u} - \boldsymbol{u}^T\boldsymbol{F},$$

where $\boldsymbol{K}$ is an N-by-N matrix and $\boldsymbol{F}$ is a size N vector. We will assume here and throughout that $\boldsymbol{K}$ is a symmetric – $\boldsymbol{K} = \boldsymbol{K}^T$.

Example: Spring with end load

Consider a spring (spring constant k), fixed at one end and subject to a dead-load F at the other end. This is an example of the case for $N = 1$. Denote the displacement at the loadd end by u. The potential energy of the spring and potential energy due to the load are:

$$\begin{aligned}\Pi_{\text{spring}} &= \frac{1}{2}uku,\\ \Pi_{\text{load}} &= -uF,\end{aligned}$$

and thus:

$$\Pi_{\text{total}} = \frac{1}{2}uku - uF\ .$$

Example: Bar with two loads

Consider the mechanical system consiting of an elastic bar of length L fixed at $x = 0$ and subject to two load, F_1 at the point $x = a$, and F_2 at the end $x = L$. Denote the displacement at $x = a$ as u_1 and the displacement at $x = L$ as u_2. Employing the information that the displacement is linear between the loads, one obtains the following expression for the potential energies:

$$
\begin{aligned}
\Pi_{\text{bar}} &= \frac{1}{2}\frac{EA}{a}u_1^2 + \frac{1}{2}\frac{EA}{L-a}\left(u_2 - u_1\right)^2 , \\
\Pi_{\text{load}} &= -F_1 u_1 - F_2 u_2 .
\end{aligned}
$$

After some manipulation, the total potential energy can be expressed as,

$$
\begin{aligned}
\Pi_{\text{total}} &= \frac{1}{2}\begin{bmatrix} u_1 & u_2 \end{bmatrix}\begin{bmatrix} \frac{EA}{a} + \frac{EA}{L-a} & -\frac{EA}{L-a} \\ -\frac{EA}{L-a} & \frac{EA}{L-a} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} u_1 & u_2 \end{bmatrix}\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \\
&= \frac{1}{2}\boldsymbol{u}^T\boldsymbol{K}\boldsymbol{u} - \boldsymbol{u}^T\boldsymbol{F} .
\end{aligned}
$$

4 Solutions for linear mechanical systems

The behavior of the solutions for linear mechanical systems can be understood by looking at the properties of the matrix $\boldsymbol{K}$. Here we will assume again that $\boldsymbol{K}$ is symmetric.

- $\boldsymbol{K}$ is symmetric positive definite (all eigenvalues are positive):
  - There is one stationary point which is also a minimum.
  - The system is stable.
- $\boldsymbol{K}$ has both positive and negative eigenvalues :
  - There is one stationary point which is a saddle point (not a minimum).
  - The system is unstable.
- $\boldsymbol{K}$ has a zero eigenvalue:
  - There may be no stationary points or there maybe multiple.
  - The system is unstable.

5 Exercise

5.1 Download files

Download the files `plotenergy1v.m` and `plotenergy2v.m` from the Lab 6 folder on bspace.

5.2 Potential energy for a 1 degree of freedom case

The function `plotenergy1v.m` plots the total potential energy for a 1 degree of freedom linear mechanical system. The potential energy for this system is,

$$\Pi_{\text{total}} = \frac{1}{2}uKu - uF \ ,$$

where K and F are scalars. This expression corresponds to the total potential energy for a spring (spring constant K) with end load F.

You should be able to run the function with the following lines,

```
>> K = 1;
>> F = 1;
>> param.u_range = [-2,2];
>> param.e_range = [-2,2];
>> plotenergy1v(K,F,param);
```

to obtain the Figure 1.

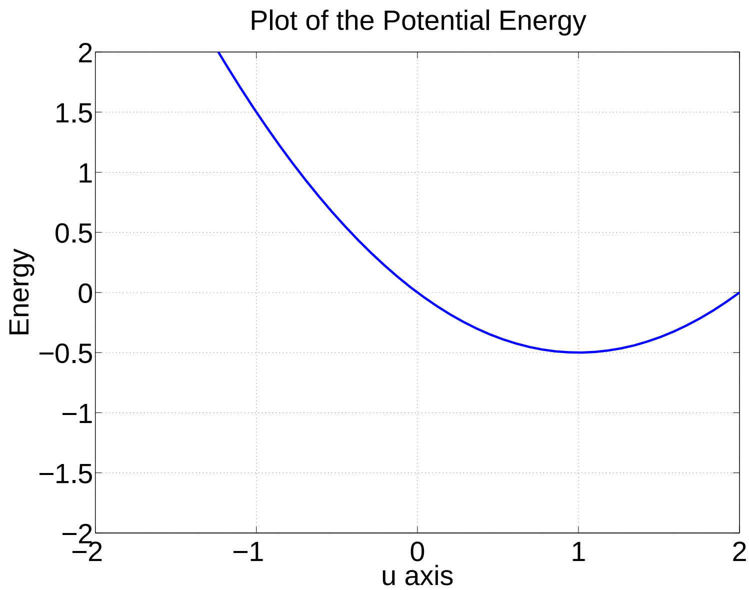


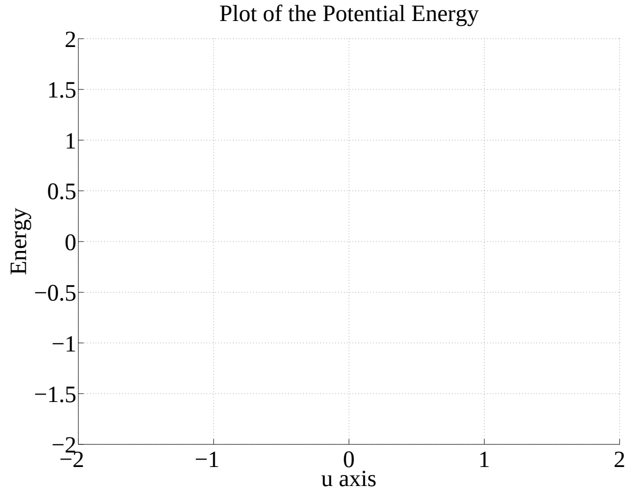
Figure 1: Sample potential energy for single degree of freedom system

1. For each of the following cases, plot the energy, and sketch the result on the axes given below (make sure key features are identified) and answer the following questions:

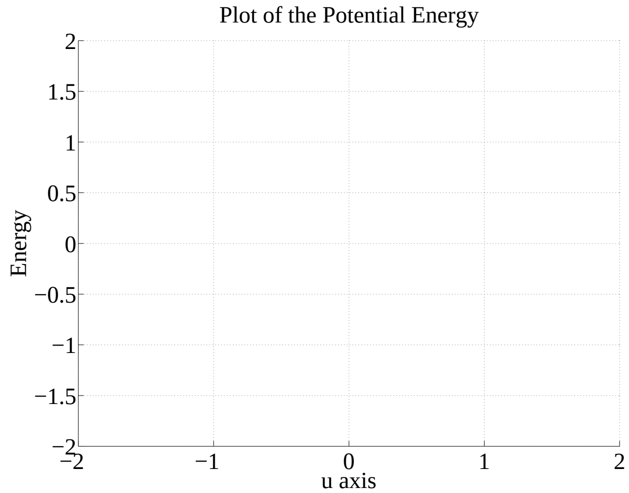
   - Does the system have a solution (equilibrium point)?

- Does the potential energy have a stationary point?
- If the system has a solution calculate it by hand. What (and where) does this solution correspond to in the plot. Is the solution a maximum, minimum, or saddle state at this point?
- If the system has an equilibrium point, is this point stable or unstable?

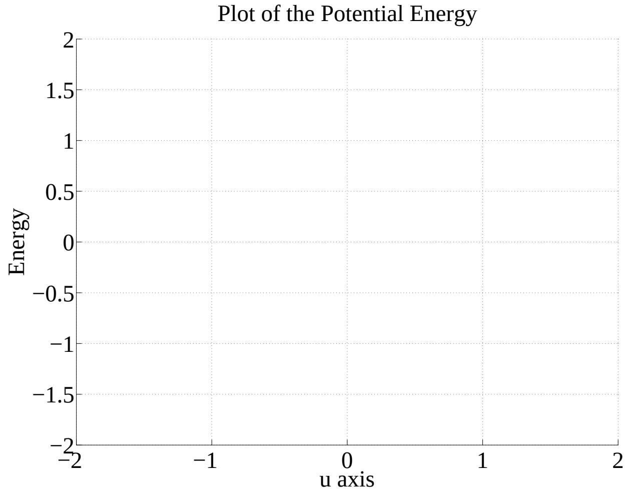
(a) $K = 2, F = 1$:



(b) $K = -2, F = 1$



(c) $K = 0, F = 1$


Plot of the Potential Energy
Energy
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2
−1
0
1
2
u axis

5.3 Potential energy for a 2 degree of freedom system

The function `plotenergy2v.m` plots the total potential energy for a 2 degree of freedom linear mechanical system as well as its countour plot and gradient. The potential energy for this system is,

$$\Pi_{\mathrm{total}} \;=\; \frac{1}{2}\boldsymbol{u}^T\boldsymbol{K}\boldsymbol{u} - \boldsymbol{u}^T\boldsymbol{F} \; ,$$

where $\boldsymbol{K}$ is a 2-by-2 matrix and $\boldsymbol{F}$ is a 2-by-1 vector.

1. **Potential energy for a 2 spring system**

   (a) Derive the expression for the total potential energy of the system constructed from 2 springs shown in Figure 2. The stiffness of the springs are k_1 and k_2, the displacement and load at the two nodes are u_1, F_1, u_2, F_2. Write down the expression for $\boldsymbol{K}$ for this system.

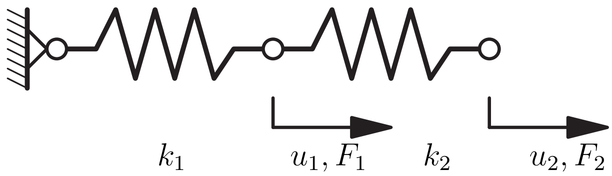


Figure 2: 2 degree of freedom system

   (b) For each of the following cases, plot the energy using the function `plotenergy2v.m` and answer the questions,

   - Does the system have a solution (equilibrium point)?
   - Does the potential energy have a stationary point?
   - If the system has a solution calculate it by hand. What does this solution correspond to in the plot. Is the solution a maximum, minimum, or a saddle state at this point?
   - If the system has an equilibrium point, is this point stable or unstable? (HINT: Compute the eigenvalues of $\boldsymbol{K}$.)

   You should be able to run the code with the following lines,

```
>> % K = DEFINE AN APPROPRIATE 2-BY-2 MATRIX;
>> % F = DEFINE AN APPROPRIATE LOAD VECTOR;
>> param.u1_range = [-2,2]; % Adjust as needed
>> param.u2_range = [-2,2]; % Adjust as needed
>> plotenergy2v(K,F,param);
```

i. $k_1 = k_2 = 1, F_1 = 0, F_2 = 1$:

ii. $k_1 = -1, k_2 = 1, F_1 = 1, F_2 = 1$:

iii. $k_1 = 0, k_2 = 1, F_1 = 1, F_2 = 1$:

2. **Potential energy for a shallow truss structure** The potential energy for the shallow truss structure shown in Figure 3 is given as,

$$\Pi_{\text{total}} \quad = \quad \frac{1}{2}\boldsymbol{u}^T\boldsymbol{K}\boldsymbol{u} - \boldsymbol{u}^T\boldsymbol{F} \ ,$$

where,

$$\boldsymbol{K} = \frac{2EA}{(1+a^2)^{3/2}}\begin{bmatrix} 1 & 0 \\ 0 & a^2 \end{bmatrix} \ .$$

For this problem assume $EA = 1$ and $F_1 = 0, F_2 = 1$. Comment on the behavior of this structure as a decreases from $a = 1$ to $a = 0$. Comment in terms of stability of the structure.

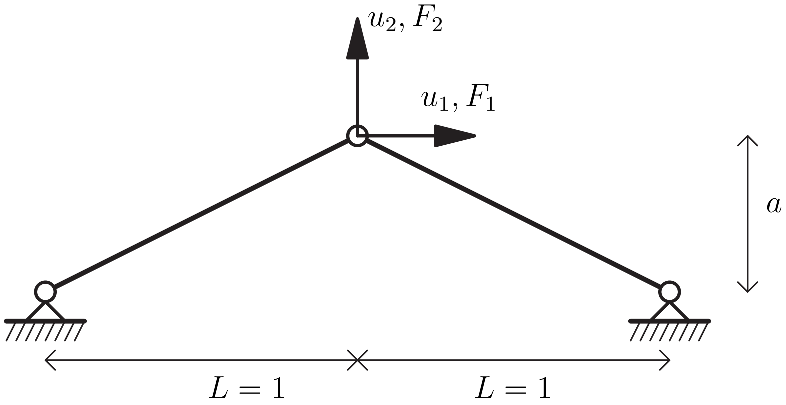


Figure 3: Shallow truss