UNIVERSITY OF CALIFORNIA AT BERKELEY
Department of Civil and Environmental Engineering
Structural Engineering, Mechanics and Materials

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Lab 9

Mechanics of Structures (CE130N)
Lab 9

1 Objective

The objective of this lab is to understand how the method of Ritz can be applied to find approximate solutions to differential equations which govern mechanical behavior. You should understand that the method of Ritz is just an application of the Principle of Minimum Potential Energy.

In this lab you will,

- Understand what it means for functions to be orthogonal.
- Use the method of Ritz to solve,
  - Tension-compression bars
  - Bending of beams

2 The Ritz method

The Ritz method can be summarized by the following steps,

1. Form the expression for the total potential energy Π_t.

$$\Pi_{\mathrm{t}} = \begin{cases} \int_0^L \frac{1}{2}E(x)I(x)(v'')^2\ dx + \Pi_{\mathrm{load,beam}} & \text{(Beam)} \\ \int_0^L \frac{1}{2}E(x)A(x)(u'')^2\ dx + \Pi_{\mathrm{load,bar}} & \text{(Tension-compression bar)} \\ \int_0^L \frac{1}{2}G(z)J(z)(\varphi'')^2\ dz + \Pi_{\mathrm{load,torsion\ bar}} & \text{(Torsion bar)} \end{cases}$$

2. Determine the form for the approximate solution,

$$v(x) \quad = \quad \sum_{i=1}^{N} c_i f_i(x)\ ,$$

where c_i are the coefficients to be determined and $f_i(x)$ are the Ritz functions. The proper selection of the Ritz functions is crucial in obtaining a good approximation. They must satisfy the following conditions,

- $f_i(x)$ must satisfy the displacement boundary conditions.

	Beam	Tension bar	Torsion bar
Order of differential equation	4	2	2
Displacement B.C.	On $v(x)$ and $\theta(x) = \frac{dv}{dx}$	On $u(x)$	On $\varphi(z)$

- $f_i(x)$ must be linearly indpendent.
- $f_i(x)$ must be nonzero for enough derivatives to make sense in the stiffness matrix.

	Beam	Tension bar	Torsion bar
Order of differential equation	4	2	2
Must be nonzero for	2 derivatives	1 derivative	1 derivative

3. Insert the approximation into the expression for the potential energy,

$$\Pi_t(v(x)) \Rightarrow \Pi_t(\mathbf{c})\ .$$

4. Apply the principle of stationary potential energy,

$$\frac{\partial \Pi_t(\mathbf{c})}{\partial \mathbf{c}} = \mathbf{Kc} - \mathbf{F} = \mathbf{0}\ ,$$

where,

$$K_{ij} \quad = \quad \begin{cases} \displaystyle\int_0^L f_i' EA f_j' dx & \text{bar} \\ \displaystyle\int_0^L f_i'' EI f_j'' dx & \text{beam} \\ \displaystyle\int_0^L f_i' GJ f_j' dx & \text{torsion bar} \end{cases} .$$

and,

$$F_i \quad = \quad \begin{cases} \displaystyle\int_0^L f_i b dx + \text{term from point loads} & \text{(Tension-compression bar)} \\ \displaystyle\int_0^L f_i q dx + \text{term from point loads} & \text{(Beam)} \\ \displaystyle\int_0^L f_i t dz + \text{term from point loads} & \text{(Torsion bar)} \end{cases} .$$

5. Solve the linear system of equations,

$$\mathbf{Kc} = \mathbf{F} \; .$$

to obtain the approximation,

$$v(x) \quad = \quad \sum_{i=1}^{N} c_i f_i(x) \; ,$$

3 Orthogonality of functions

Two functions $f(x)$ and $g(x)$ defined on the interval $[a, b]$ are orthogonal if and only if,

$$(f, g) \quad := \quad \int_a^b f(x) g(x) \; dx = 0 \; .$$

If they are orthogonal, they are linearly independent. This is similar to the situation for vectors. Given to vectors $\mathbf{a}$ and $\mathbf{b}$, if they are orthogonal $\mathbf{a} \cdot \mathbf{b}$, they are linearly independent.

4 Sample file for Ritz program for beams

```
% -- Example of a uniform cantilever beam (length L=1, stiffness EI=1)
%    loaded at the end (x=L) with a point load P=1.
%
%    The exact solution has the form of a cubic function with
%
%        v(x==L) = PLˆ3/(3EI)
%
clear all;

% -- Interval on which the beam is defined
%    Assume the beam is length of 1
%
a = 0;
b = 1;
ritz.interval = [a,b];

% -- Order of the differential equation
%    The differential equation for a beam is 4th order
%
ritz.order =  4;

% -- The functions in the Ritz method
%    The cell array ’ritz.f’ holds the function handles to
%    the Ritz functions and its derivatives. Assume we have
%
%      N           -- Number of Ritz functions
%      ritz.order  -- Order of differential equation
%
%    Then we require expressions up to the ’M’th derivative of the functions
%
%      M = ritz.order/2+1;
%
%    Thus the size of ’ritz.f’ is M-by-N, where
%
%      -Rows    correspond to the order of derivative
%      -Columns correspond to the different functions
%
%    For example,
%
%      ritz.f{i,j}  -- Function handle to the (i-1)st derivative of the
%                                              j   th Ritz function
%
ritz.N  = 1;                            % -- Number of functions
ritz.f  = cell(ritz.order/2+1,ritz.N);  % -- Cell array to hold functions
                                        %    and their derivatives

% -- Use polynomial approximation as Ritz functions
%
%    The 1st Ritz function is a quadratic function xˆ2
ritz.f{1,1}  = @(z)z.ˆ2;                        % -- 0th derivative
ritz.f{2,1}  = @(z)2*z;                         % -- 1st derivative
ritz.f{3,1}  = @(z)2*ones(size(z,1),size(z,2)); % -- 2nd derivative

%    The 2nd Ritz function is a cubic function xˆ3
%    COMPLETE THIS PROCEDURE


%    The 3rd Ritz function is a quartic function x^4
%    COMPLETE THIS PROCEDURE


% -- Define effective material
%    For the beam the effective bending rigidity EI must be defined
%
E          = 1;
I          = 1;
ritz.mat   = E*I;

% -- Define distributed load
%    For the beam the distributed load q must be defined
%
ritz.distload = 0;

% -- Define point loads (forces and moments)
%    ritz.pointload= [cd1,  cd2, ... , cd_np
%                      x1,   x2, ... ,  x_np;
%                    val1, val2, ... ,val_np;];
%    Thus the,
%      1st row : Type of loading
%      2nd row : Location of loading
%      3rd row : Value of the loading
%
%    The ’cd’ denotes the type of load
%
%      cd==0  : Point load   (P)
%      cd==1  : Point moment (M)
%
ritz.pointload = [  0;
                    b;
                    1;];

% -- Define displacement BC (displacement and rotation)
%    ritz.dispbc = [cd1,  cd2, ... , cd_np
%                    x1,   x2, ... ,  x_np;
%                  val1, val2, ... ,val_np;];
%    Thus the,
%      1st row : Type of displacement BC
%      2nd row : Location of displacement BC
%      3rd row : Value of the displacement BC
%
%    The ’cd’ denotes the type of displacement BC
%
%      cd==0  : On the displacement (u)
%      cd==1  : On the rotation     (du/dx)
%
% -- Define displacement BC
%
ritz.dispbc= [ 0, 1;
               a, a;
               0, 0;];
```

5 Exercise

5.1 Download files

1. Download the file `ritz.zip` into your `ce130n/programs` directory and unzip it.
2. Go to the `ce130n/programs/ritz/exercise/` directory, and execute the file `init.m`. This will set the necessary paths to run the files.

YOU MUST RUN THE FILE init.m EVERYTIME YOU START UP MATLAB.

5.2 Function handles and plotting functions

In this section you will learn how to define very simple functions within MATLAB and visually plot the functions.

5.2.1 Define, plot, and integrate a function

You have already seen how you can define complicated functions by creating seperate files. Here, you will understand how you can create small simple functions without saving them in seperate files. By writing the following line of code,

```
>> f = @(z)z.ˆ2;              % -- f is a function where f(z) = zˆ2;
```

one can define a function `f` which given an argument `z` returns z^2. Confirm this by the line of code,

```
>> f(3)
```

which should return `9`. One can easily plot this function on the interval $[0,2]$ by the line of code,

```
>> fplot(f,[0,2]);     % -- Plot f on the interval [0,2]
```

The function `fplot` automatically generates the plot for the function on the given interval. One can also integrate the function `f` on the interval $[0,2]$ by the line of code,

```
>> quad(f,0,2)     % -- Integrate f on the interval [0,2]
```

which should return `8/3`.

5.2.2 Define a function which is the product of two functions

One can also define a function which is the product of two functions by the following lines of code,

```
>> f = @(z)z.ˆ2;              % -- f  is a function where f(z) = zˆ2;
>> g = @(z)z.ˆ3;              % -- g  is a function where g(z) = zˆ3;
>> fg= @(z)func_prod(z,f,g);  % -- fg is a function where fg(z)= f(z)*g(z)
                              %                                = zˆ5;
```

Here `fg` is the function z^5. Confirm this by plotting the function `fg` using `fplot` or evaluating `fg(3)`. Using this method one can easily find the integral of the product of two functions by using `quad`,

```
>> quad(fg,0,2)     % -- Integrate f-times-g on the interval [0,2]
```

This should return $32/3$.

5.2.3 Plotting polynomial functions

One can also use the following function to produce polynomials.

```
>> order = 3;                    % -- Order of the polynomial
>> dorder= 0;                    % -- Order of derivatives to take
>> a     = -1;                   % -- Left  value of interval
>> b     =  1;                   % -- Right value of interval
>> f     = @(z)polynomial(z,order,dorder,a,b);
```

The function `f` is the standard polynomial of order 3 scaled and translated so that $f(a) = 0$ and $f(b) = 1$. Thus,

$$f(z) = \left(\frac{x-a}{b-a}\right)^3 \ .$$

Confirm this by typing `fplot(f,[a,b])`. If one desires the 2nd derivative of this function,

$$g(z) = \frac{d^2 f}{dz^2} = \frac{6}{(b-a)^2}\left(\frac{x-a}{b-a}\right) \ ,$$

one can obtain this by,

```
>> order = 3;                    % -- Order of the polynomial
>> dorder= 2;                    % -- Order of derivatives to take
>> a     = -1;                   % -- Left  value of interval
>> b     =  1;                   % -- Right value of interval
>> g     = @(z)polynomial(z,order,dorder,a,b);
```

where the variable `dorder` has been set to 2. Confirm that `g(z)` is linear by plotting it, `fplot(g,[a,b])`.

5.2.4 Plotting Legendre polynomials

One can use the following function to produce Legendre polynomials $P_n(x)$.

```
>> order = 3;                    % -- Order of the polynomial
>> dorder= 0;                    % -- Order of derivatives to take
>> a     = -1;                   % -- Left  value of interval
>> b     =  1;                   % -- Right value of interval
>> f     = @(z)legendre_polynomial(z,order,dorder,a,b);
```

The function `f` is the Legendre polynomial of order 3 scaled and translated so that $f(a) = -1$ and $f(b) = 1$. Thus,

$$f(z) = P_3\left(\frac{x-a}{b-a} - 1\right) \ .$$

Confirm this by typing `fplot(f,[a,b])`. If one desires the 1st derivative of this function,

$$g(z) = \frac{df}{dz} \ .$$

one can obtain this by,

```
>> order = 3;                    % -- Order of the polynomial
>> dorder= 1;                    % -- Order of derivatives to take
>> a     = -1;                   % -- Left  value of interval
>> b     =  1;                   % -- Right value of interval
>> g     = @(z)legendre_polynomial(z,order,dorder,a,b);
```

where the variable `dorder` has been set to 1. Confirm the shape of `g(z)` by plotting it, `fplot(g,[a,b])`.

5.2.5 Plotting sine functions

One can use the following function to produce sine functions which are zero at the end of the interval $[a, b]$.

```
>> order = 3;                 % -- Order of the polynomial
>> dorder= 0;                 % -- Order of derivatives to take
>> a     = -1;                % -- Left  value of interval
>> b     =  1;                % -- Right value of interval
>> f     = @(z)sine_dd(z,order,dorder,a,b);
```

The function `f` is the sine function of order 3 scaled and translated so that $f(a) = 0$ and $f(b) = 0$. Thus,

$$f(z) = \sin\left(3\pi\frac{x-a}{b-a}\right) \ .$$

Confirm this by typing `fplot(f,[a,b])`. If one desires the 1st derivative of this function,

$$g(z) = \frac{df}{dz} \ .$$

one can obtain this by,

```
>> order = 3;                 % -- Order of the polynomial
>> dorder= 1;                 % -- Order of derivatives to take
>> a     = -1;                % -- Left  value of interval
>> b     =  1;                % -- Right value of interval
>> g     = @(z)sine_dd(z,order,dorder,a,b);
```

where the variable `dorder` has been set to 1. Confirm the shape of `g(z)` by plotting it, `fplot(g,[a,b])`.

5.2.6 Plotting and integrating functions

- Plot a polynomial term of order 3 and 4 on the interval $[1, 2]$ using the function `polynomial.m` and draw them in the paper. The 3rd-order polynomial is defined as,

  $$f(x) = \left(\frac{x-1}{2-1}\right)^3 ,$$

  and the 4th-order polynomial as,

  $$g(x) = \left(\frac{x-1}{2-1}\right)^4 .$$

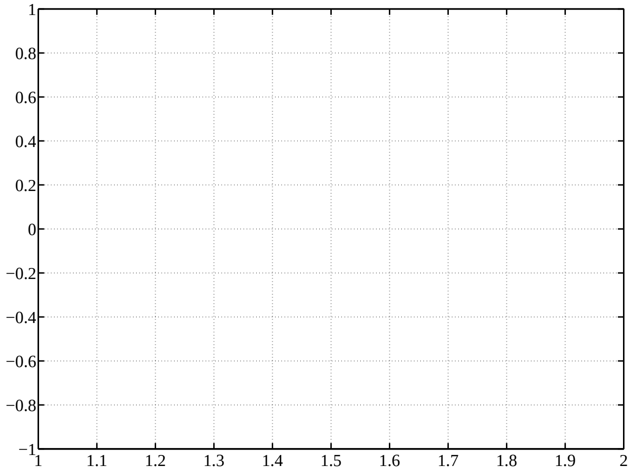
  Observe that they both take the value of 0 at the left end point and 1 at the right end point.

  What can you say about the shape of the functions as you increase the order of the polynomial? Do they look more alike or are they still different? What does this tell you about the linear independence of the functions?

  Evaluate the expression,

  $$\int_1^2 f(x)g(x)\, dx .$$

  This can be done using the method introduced earlier using function handles and the function `func_prod.m`. What is the value you obtain? Are the functions orthogonal? Are any two polynomial functions of different order orthogonal?



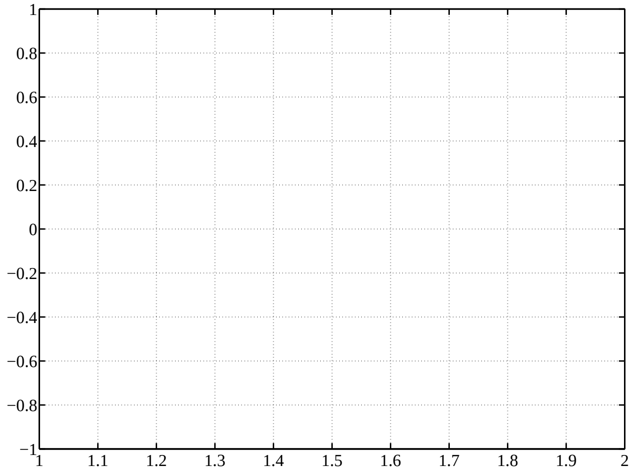
- Plot a Legendre polynomial of order 3 and 4 on the interval $[1, 2]$ using the function `legendre_polynomial.m` and draw them in the paper. Define $f(z)$ as the 3rd-order Legendre polynomial and $g(z)$ as the 4th-order Legendre polynomial. Observe that they both take the value of -1 at the left end point and 1 at the right end point.

  What can you say about the shape of the functions as you increase the order of the polynomial? Do they look more alike or are they still different? What does this tell you about the linear independence of the functions?

  Evaluate the expression,

  $$\int_1^2 f(x)g(x)\;dx\;.$$

  This can be done using the method introduced earlier using function handles and the function `func_prod.m`. What is the value you obtain? Are the functions orthogonal? Are any two polynomial functions of different orthogonal?

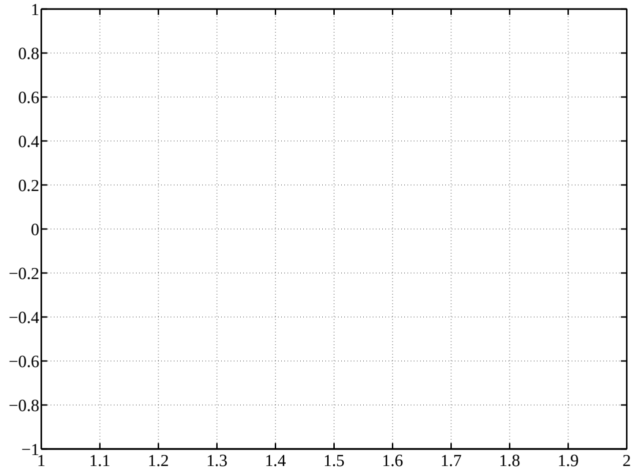


- Plot a sine function of order 3 and 4 on the interval $[1, 2]$ using the function `sine_dd.m` and draw them in the paper. Define $f(z)$ as the 3rd-order sine function and $g(z)$ as the 4th-order sine function. Observe that they both take the value of 0 at the left end point and 0 at the right end point.

  Evaluate the expression,

  $$\int_1^2 f(x)g(x)\ dx\ .$$

  This can be done using the method introduced earlier using function handles and the function `func_prod.m`. What is the value you obtain? What can you say about the shape of the functions as you increase the order? Do they look more alike or are they still different? What does this tell you about the linear independence of the functions?



5.3 The method of Ritz

Let us use the method of Ritz to solve some mechanical problems. The method requires assuming a form for the approximate solution,

$$v(x) \quad = \quad \sum_{i=1}^{N} c_i f_i(x),$$

where the c_i are the coefficients to be determined and $f_i(x)$ are the Ritz functions. To successfully apply the method of Ritz, one must select Ritz functions which satisfy the boundary conditions. (One can use functions which do not satisfy the boundary conditions, but it requires additional treatment.) Thus, depending on the boundary conditions of the problem one must change the Ritz functions. Here we introduce the Ritz functions and corresponding family of mechanical problems they can be used to solve.

1. Using polynomial term functions defined on the interval $[a, b]$,

   $$f_n(x) = \left(\frac{x-a}{b-a}\right)^n ,$$

   which satisfy the boundary condition,

   $$\begin{aligned} f_n(a) &= 0, \\ f_n(b) &= 1 . \end{aligned}$$

   These are suited in solving,

   - Tension-compression bar with one end fixed.
   - Cantilever beams.
   - Torsion bars with one end fixed.

2. Using sine functions defined on the interval $[a, b]$,

   $$f_n(x) = \sin\left(n\pi \frac{x-a}{b-a}\right),$$

   which satisfy the boundary condition,

   $$\begin{aligned} f_n(a) &= 0, \\ f_n(b) &= 0 . \end{aligned}$$

   These are suited in solving,

   - Tension-compression bar with both ends fixed.
   - Simply supported beams.
   - Torsion bars with both ends fixed.

The general structure to solve the problem involves the following MATLAB commands,

```
>> ritz_data_beam_exercise1;  % -- Load data structure for problem
>> plotproblem(ritz);         % -- Plots configuration of problem
>> plotritzf(ritz,[1]);       % -- Plots Ritz function for numbers [1]
>> solveritz;                 % -- Solves problem
                              %    Forms K,F and solve for c=K\F
>> plotsol(ritz,c);           % -- Plots solution and components of solution
```

You can obtain the displacement at any location along the beam $[0, L]$ by the function,

```
>> xloc = 1.0;                 % -- Location to obtain approximate displacement
>> evaldisp(ritz,c,xloc)       % -- Use ritz structure and approximation c
```

5.3.1 Polynomials for the cantilever beam

Here we use polynomials to solve the cantilever beam.

1. Run the problem using polynomials.

```
>> ritz_data_beam_exercise1;  % -- Load data structure for problem
>> plotproblem(ritz);         % -- Plots configuration of problem
>> plotritzf(ritz,[1]);       % -- Plots Ritz function numbers [1]
>> solveritz;                 % -- Solves problem
                              %    Forms K,F and solve for c=K\F
>> plotsol(ritz,c);           % -- Plots solution and components of solution
```

This problem solves for a uniform cantilever ($EI = 1, L = 1$) with an end vertical load ($P = 1$). The Ritz approximation is,

$$\begin{aligned} v(x) &= \sum_{i=1}^{1} c_i f_i(x), \\ f_1(z) &= z^2 , \end{aligned}$$

using only one quadratic function. Note that this function $f_1(x)$ satisfies the displacement boundary conditions, zero deflection $f_1(x = 0) = 0$ and zero rotation $f_1'(x = 0) = 0$ at $x = 0$.

What is the displacement at $x = L$? How does it compare with the exact solution, $v(x = L) = PL^3/3EI$?

2. Modify the file `ritz_data_beam_exercise1.m` so that you use two functions in your Ritz approximation,

$$\begin{aligned} v(x) &= \sum_{i=1}^{2} c_i f_i(x), \\ f_1(z) &= z^2\,, \\ f_2(z) &= z^3\,. \end{aligned}$$

To do this you will have to define functions in the cell array `ritz.f` for the cubic function and its derivatives up to 2. Observe how this is done for the quadratic function.

What is the displacement at $x = L$? How does it compare with the exact solution, $v(x = L) = PL^3/3EI$?

3. Modify the file `ritz_data_beam_exercise1.m` so that you use 3 functions in your Ritz approximation,

$$\begin{aligned} v(x) &= \sum_{i=1}^{3} c_i f_i(x), \\ f_1(z) &= z^2\,, \\ f_2(z) &= z^3\,. \\ f_3(z) &= z^4\,. \end{aligned}$$

To do this you will have to define functions in the cell array `ritz.f` for the quartic function and its derivatives up to 2.

What is the displacement at $x = L$? How does it compare with the exact solution, $v(x = L) = PL^3/3EI$? Does this 3 term approximation give you better results than the 2 term approximation? Explain why or why not.

4. Modify the file `ritz_data_beam_exercise1.m` so that you use 2 functions in your Ritz approximation,

$$\begin{aligned} v(x) &= \sum_{i=1}^{2} c_i f_i(x), \\ f_1(z) &= z^2 \,, \\ f_2(z) &= z^2 \,. \end{aligned}$$

Does MATLAB give you some complaint in running this problem? Does this 2 term approximation give you better results than the 1 term approximation with 1 quadratic function? Explain why or why not. Why do you think MATLAB complains? (HINT: Look at the **K** matrix).

5. Modify the file `ritz_data_beam_exercise1.m` so that you use 2 functions in your Ritz approximation,

$$\begin{aligned} v(x) &= \sum_{i=1}^{2} c_i f_i(x), \\ f_1(z) &= z \,, \\ f_2(z) &= z^2 \,. \end{aligned}$$

Does $f_1(x)$ satisfy the displacement boundary conditions for this problem? Does MATLAB give you some complaint in running this problem? Does this 2 term approximation give you better results than the 1 term approximation with 1 quadratic function? Explain why or why not. Why do you think MATLAB complains? (HINT: Look at the **K** matrix).

6. Modify the file `ritz_data_beam_exercise1.m` so that you use two functions in your Ritz approximation,

$$\begin{aligned} v(x) &= \sum_{i=1}^{2} c_i f_i(x), \\ f_1(z) &= z^2 \,, \\ f_2(z) &= (z-1)^3 \,. \end{aligned}$$

Does $f_2(x)$ satisfy the displacement boundary conditions for this problem? Does MATLAB give you some complaint in running this problem? Does this 2 term approximation give you better results than the 1 term approximation with 1 quadratic function? Explain why or why not.

7. Run the file `ritz_data_beam_exercise2.m` which solves this problem using up to N terms.

$$\begin{aligned} v(x) &= \sum_{i=1}^{N} c_i f_i(x), \\ f_i(z) &= z^{i+1} \,. \end{aligned}$$

Confirm that the results you obtain for $N = 1, 2, 3$ coincide with your previous results.

5.3.2 Polynomials for the cantilever beam

Here we use polynomials to solve the tension-compression bar with one end fixed.

1. Run the file `ritz_data_bar_exercise3.m` which solves the problem introduced in the lecture for the tension-compression bar. Confirm that the results coincide. Try varying N.

5.3.3 Sine functions for the simply supported beam

Here we use sine functions to solve the simply supported beam.

1. Run the file `ritz_data_beam_exercise4.m`

   This problem solves for a uniform simply supported beam ($EI = 1, L = 1$) with a uniform distributed load ($q = 1$). The Ritz approximation is,

$$\begin{aligned} v(x) &= \sum_{i=1}^{N} c_i f_i(x), \\ f_n(x) &= \sin\left(n\pi\frac{x-a}{b-a}\right) . \end{aligned}$$

   Note that this functions $f_i(x)$ satisfies the displacement boundary conditions, zero deflection $f_i(x = 0) = 0$ and $f_i(x = L) = 0$ at $x = 0$ and $x = L$.

   What is the displacement at $x = L/2$? How does it compare with the exact solution, $v(x = L/2) = 5qL^4/384EI$? Fill in the table for the middle displacement with respect to N. Does the approximation get better? Do the $N = \text{even}$ terms contribute to a better solution? Compute the relative error.

N	1	2	3	4	5	6
v(x=L/2)						
relative error						

5.3.4 Sine functions for the doubly fixed tension-compression bar

Here we use sine functions to solve the tension-compression bar with both ends fixed.

1. Run the file `ritz_data_bar_exercise5.m`

   This problem solves for a uniform bar ($EA = 1, L = 1$) with a varying distributed load ($b = \sin\left(2\pi \frac{x}{L}\right)$). You have solved this problem in Lab 3. The Ritz approximation is,

$$\begin{aligned} v(x) &= \sum_{i=1}^{N} c_i f_i(x), \\ f_n(x) &= \sin\left(n\pi \frac{x-a}{b-a}\right) \ . \end{aligned}$$

   Note that this functions $f_i(x)$ satisfies the displacement boundary conditions, zero deflection $f_i(x = 0) = 0$ and $f_i(x = L) = 0$ at $x = 0$ and $x = L$.

   What is the displacement at $x = 3L/4$? How does it compare with the exact solution?

$$v(x) = \left(\frac{L}{2\pi}\right)^2 \sin\left(2\pi \frac{x}{L}\right) \ .$$

   Fill in the table for the displacement with respect to N. Does the approximation get better? Why or why not? Compute the relative error.

N	1	2	3	4
v(x=3L/4)				
relative error				