

## **Mechanics of Structures (CE130N)**

### **Lab 7**

#### **1 Objective**

The objective of this lab is to grasp a pictorial understanding of the principle of stationary potential energy through the visualization of the potential energy in 1,2, or 3 variables.

In this lab you will,

- Use a given function to investigate the behavior of the potential energy of mechanical systems with 1, 2, or 3 variables.

## 2 Principle of stationary potential energy

Define the following quantities,

- $\Pi_{\text{total}}$  : Total potential energy of the mechanical system,
- $\Pi_{\text{elastic}}$  : Elastic energy in the mechanical system,
- $\Pi_{\text{load}}$  : Energy due to the load,

and define,

$$\Pi_{\text{total}} := \Pi_{\text{elastic}} + \Pi_{\text{load}} .$$

Assume that all the energy quantities noted above depend on  $N$  variables or displacements,  $u_1, \dots, u_N$ , i.e.,

$$\Pi_{\text{total}}(u_1, \dots, u_N) .$$

By defining the vector  $\mathbf{u}$ ,

$$\mathbf{u} := \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} ,$$

we can denote the dependence as,

$$\Pi_{\text{total}}(\mathbf{u}) .$$

The principle of stationary potential energy states,

The mechanical system is in equilibrium  $\Leftrightarrow \Pi_{\text{total}}$  is stationary.

More concretely this implies the following,

The mechanical system is in equilibrium at  $\hat{\mathbf{u}} \Leftrightarrow \frac{\partial \Pi_{\text{total}}}{\partial u_i}(\hat{\mathbf{u}}) = 0$  for  $i = 1, \dots, N$

This principle allows us to look for the states of equilibrium of the mechanical system by looking for the stationary points of the potential energy. This principle also tells us if that there are no stationary points, then there are no states of equilibrium.

## 3 Potential energy for linear mechanical systems

For the case of linear mechanical systems,  $\Pi_{\text{total}}$  can always be written as,

$$\Pi_{\text{total}} = \frac{1}{2} \mathbf{u}^T \mathbf{K} \mathbf{u} - \mathbf{u}^T \mathbf{F} ,$$

where  $\mathbf{K}$  is a  $N$ -by- $N$  matrix and  $\mathbf{F}$  is a size  $N$  vector. We assume here that  $\mathbf{K}$  is a symmetric matrix.

### Example: Spring with end load

The mechanical system with a spring (spring constant  $k$ ) with load  $F$  attached to the end is an example of the case for  $N = 1$ . Denote the displacement at the end as  $u$ . The elastic energy and energy due to the load are,

$$\begin{aligned}\Pi_{\text{total}} &= \frac{1}{2}uku, \\ \Pi_{\text{load}} &= -uF,\end{aligned}$$

and thus,

$$\Pi_{\text{total}} = \frac{1}{2}uku - uF.$$

### Example: Bar with middle and end load, with assumption of linear displacements

Consider the mechanical system of a bar of length  $L$  fixed at  $x = 0$  with two loads,  $F_1$  at the point  $x = a$  and  $F_2$  at the end  $x = L$ , introduced in class. Denote the displacement at  $x = a$  as  $u_1$  and the displacement at  $x = L$  as  $u_2$ . Employing the information that the displacement is linear between the loads, one obtains the following expression for the energies,

$$\begin{aligned}\Pi_{\text{total}} &= \frac{1}{2} \frac{EA}{a} u_1^2 + \frac{1}{2} \frac{EA}{L-a} (u_2 - u_1)^2, \\ \Pi_{\text{load}} &= -F_1 u_1 - F_2 u_2.\end{aligned}$$

With some manipulation, the total potential energy can be expressed as,

$$\begin{aligned}\Pi_{\text{total}} &= \frac{1}{2} \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} \frac{EA}{a} + \frac{EA}{L-a} & -\frac{EA}{L-a} \\ -\frac{EA}{L-a} & \frac{EA}{L-a} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \\ &= \frac{1}{2} \mathbf{u}^T \mathbf{K} \mathbf{u} - \mathbf{u}^T \mathbf{F}.\end{aligned}$$

## 4 Solutions for linear mechanical systems

The behavior of the solutions for linear mechanical systems can be understood by looking at the properties of the matrix  $\mathbf{K}$ . Here we will assume again that  $\mathbf{K}$  is symmetric.

- $\mathbf{K}$  is symmetric positive definite (all eigenvalues are positive):
  - There is one stationary point which is also a minimum.
  - The system is stable.
- $\mathbf{K}$  has both positive and negative eigenvalues :
  - There is one stationary point which is a saddle point (not a minimum).
  - The system is unstable.
- $\mathbf{K}$  has a zero eigenvalue:
  - There may be no stationary points or there maybe multiple.
  - The system is unstable.

## 5 Exercise

### 5.1 Download files

1. Download the file `matlab.zip` into your `ce130n/week7/lab/` directory and unzip it.

### 5.2 Potential energy for a 1 variable case

Use the function `plotenergy1v.m` which plots the total potential energy for a 1 variable linear mechanical system. The potential energy for this system is,

$$\Pi_{\text{total}} = \frac{1}{2}uKu - uF,$$

where  $K$  and  $F$  are scalars. This expression corresponds to the total potential energy for a spring (spring constant  $K$ ) with end load  $F$ .

You should be able to run the function with the following lines,

```
>> K = 1;  
>> F = 1;  
>> param.u_range = [-2,2];  
>> param.e_range = [-2,2];  
>> plotenergy1v(K,F,param);
```

to obtain the Figure 1.

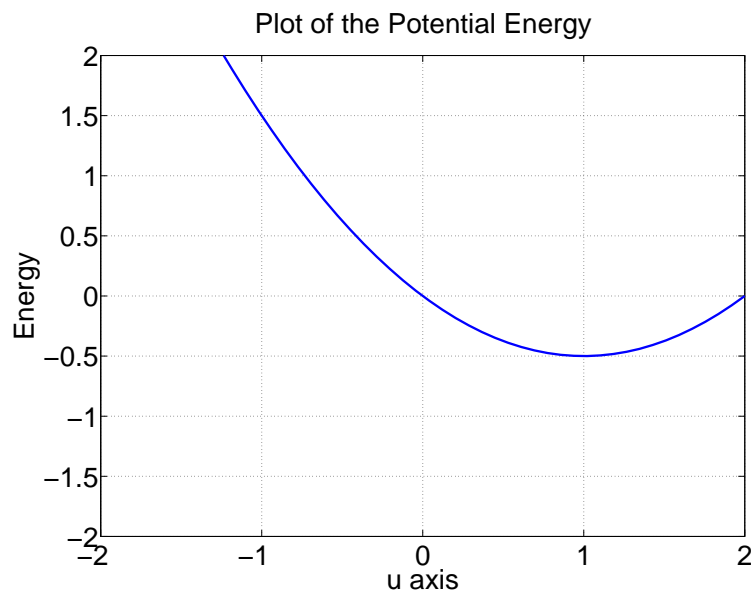
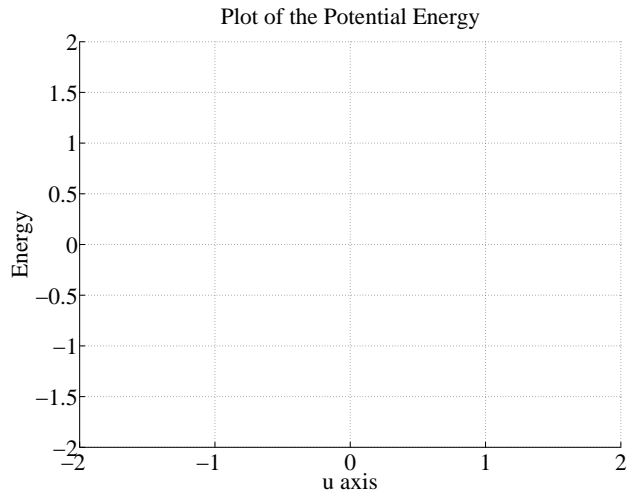


Figure 1: Sample potential energy in 1 variable

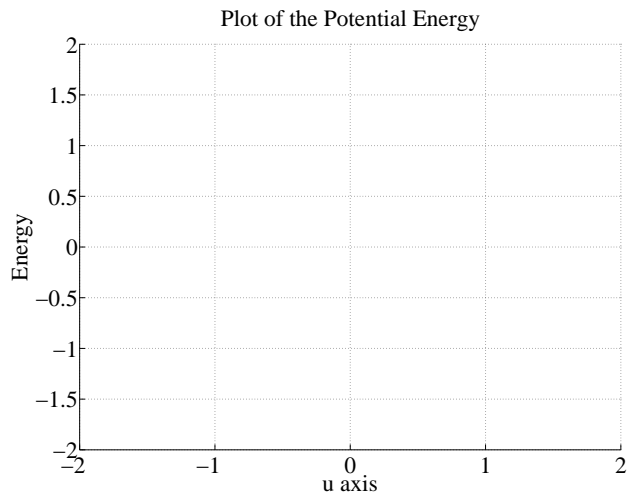
1. For each of the following cases, plot the energy and draw them in the figure (make sure key features are identified) and answer the questions,
  - Does the system have a solution (equilibrium point)?
  - Does the potential energy have a stationary point?

- If the system has a solution calculate it by hand. What (and where) does this solution correspond to in the plot. Is the solution maximum or minimum at this point?
- If the system has an equilibrium point, is this point stable or unstable?

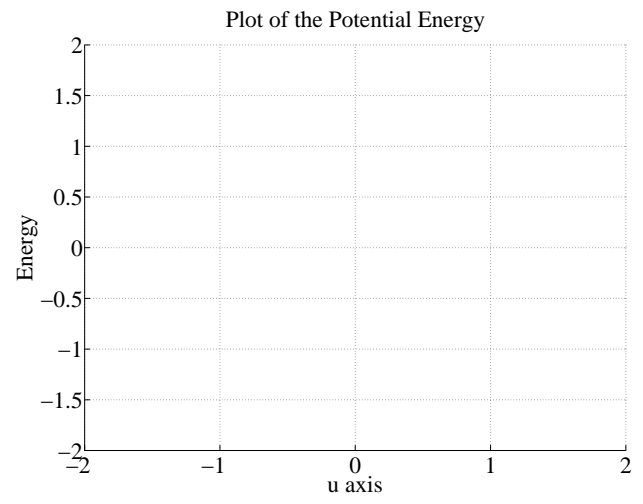
(a)  $K = 2, F = 1$ :



(b)  $K = -2, F = 1$



(c)  $K = 0, F = 1$



### 5.3 Potential energy for a 2 variable case

Use the function `plotenergy2v.m` which plots the total potential energy for a 2 variable linear mechanical system. The potential energy for this system is,

$$\Pi_{\text{total}} = \frac{1}{2} \mathbf{u}^T \mathbf{K} \mathbf{u} - \mathbf{u}^T \mathbf{F},$$

where  $\mathbf{K}$  is a 2-by-2 matrix and  $\mathbf{F}$  is a 2-by-1 vector.

#### 1. Potential energy for a 2 spring system

- (a) Derive the expression for the total potential energy of the system constructed from 2 springs shown in Figure 2. The stiffness of the springs are  $k_1$  and  $k_2$ , the displacement and load at the two nodes are  $u_1, F_1, u_2, F_2$ . Write down the expression for  $\mathbf{K}$  for this system.

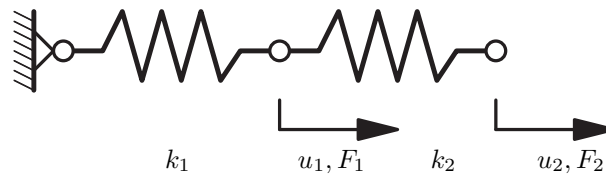


Figure 2: 2 variable system

(b) For each of the following cases, plot the energy using the function `plotenergy2v.m` and answer the questions,

- Does the system have a solution (equilibrium point)?
- Does the potential energy have a stationary point?
- If the system has a solution calculate it by hand. What does this solution correspond to in the plot. Is the solution maximum, minimum, or a saddle point at this point?
- If the system has an equilibrium point, is this point stable or unstable? (HINT: Compute the eigenvalues of  $\mathbf{K}$ .)

You should be able to run the code with the following lines,

```
>> K = DEFINE THE 2-BY-2 MATRIX;  
>> F = DEFINE THE LOAD VECTOR;  
>> param.u1_range = [-2,2];  
>> param.u2_range = [-2,2];  
>> plotenergy2v(K,F,param);
```

i.  $k_1 = k_2 = 1, F_1 = 0, F_2 = 1$ :



ii.  $k_1 = -1, k_2 = 1, F_1 = 1, F_2 = 1$ :

iii.  $k_1 = 0, k_2 = 1, F_1 = 1, F_2 = 1$ :

2. **Potential energy for a shallow truss structure** The potential energy for the shallow truss structure shown in Figure 3 is given as,

$$\Pi_{\text{total}} = \frac{1}{2} \mathbf{u}^T \mathbf{K} \mathbf{u} - \mathbf{u}^T \mathbf{F},$$

where,

$$\mathbf{K} = \frac{2EA}{(1+a^2)^{3/2}} \begin{bmatrix} 1 & 0 \\ 0 & a^2 \end{bmatrix}.$$

For this problem assume  $EA = 1$  and  $F_1 = 0, F_2 = 1$ . Comment on the behavior of this structure as  $a$  decreases from  $a = 1$  to  $a = 0$ . Comment in terms of stability of the structure.

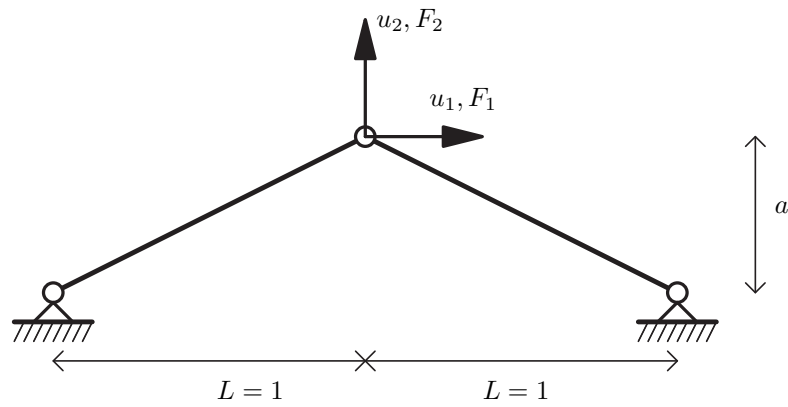


Figure 3: Shallow truss

## 5.4 Potential energy for a 3 variable case

Use the function `plotenergy3v.m` which plots the total potential energy for a 3 variable linear mechanical system. The potential energy for this system is,

$$\Pi_{\text{total}} = \frac{1}{2} \mathbf{u}^T \mathbf{K} \mathbf{u} - \mathbf{u}^T \mathbf{F},$$

where  $\mathbf{K}$  is a 3-by-3 matrix and  $\mathbf{F}$  is a 3-by-1 vector.

1. **Potential energy for a 3 spring system** The expression for the matrix  $\mathbf{K}$  of the system constructed from 3 springs shown in Figure 4 is,

$$\mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix},$$

where  $k_1$ ,  $k_2$ , and  $k_3$  are the stiffnesses of the springs.

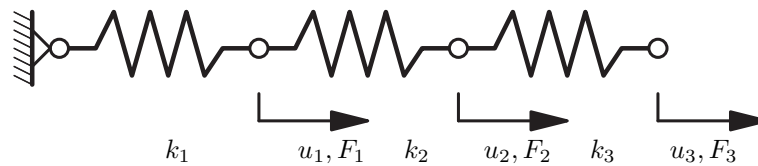


Figure 4: 3 variable system

- Using the function `plotenergy3v.m`, plot the energy for the following cases and answer the questions,
  - Does the system have a solution (equilibrium point)?
  - Does the potential energy have a stationary point?
  - If the system has a solution what does this solution correspond to in the plot. Is the solution maximum, minimum, or a saddle point at this point?
  - If the system has an equilibrium point, is this point stable or unstable? (HINT: Compute the eigenvalues of  $\mathbf{K}$ .)

You should be able to run the code with the following lines,

```
>> K = DEFINE THE 3-BY-3 MATRIX;
>> F = DEFINE THE LOAD VECTOR;
>> param.u1_range = [-2,2];
>> param.u2_range = [-2,2];
>> param.u3_range = [-2,2];
>> param.iso1_val = 1;
>> param.iso2_u3 = 0;
>> param.iso2_ulrot= 45;
>> param.iso2_u2rot= 0;
>> plotenergy3v(K,F,param);
```

i.  $k_1 = k_2 = k_3 = 1, F_1 = 0, F_2 = 1, F_3 = 1$ :

ii.  $k_1 = -1, k_2 = k_3 = 1, F_1 = 0, F_2 = 1, F_3 = 1$ :