UNIVERSITY OF CALIFORNIA AT BERKELEY
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Structural Engineering, Mechanics and Materials

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Lab 12

Mechanics of Structures (CE130N)
Lab 12

1 Objective

The objective of this lab is to see how the principle of stationary potential energy can be applied not only to mechanical problems depending on 1 variable but to mechanical problems in multi-variables. The problem of computing approximations to the torsional stiffness k_T of rectangular bars is treated. One should observe that the procedure involved in computing the approximation using the principle of stationary potential energy is no different from the previous one-dimensional problems.

2 Computing the torsional stiffness of bars

The torsional stiffness k_T is defined as

$$\begin{aligned} k_T &:= \frac{T}{\theta}, \\ \theta &:= \frac{d\varphi}{dz}, \end{aligned}$$

where T is the applied torque, φ is the twist angle, and θ is the twist angle per unit length. k_T of a solid circular bar with radius a can be obtained as,

$$\begin{aligned} k_T &= GJ, \\ J &= \int_A r^2 \, dA = \frac{\pi}{2} a^4, \end{aligned}$$

where G is the shear modulus and J is the polar moment of inertia of the cross section. This formula is derived under the assumption that cross sections which are plane remain plane after deformation. This assumption does not necessarily hold for general solid cross sections such as rectangles. To fully derive the expression for k_T for general solid cross sections, one must employ the theory of elasticity (covered in CE131). Here the result is introduced and a method employing the principle of stationary potential energy is used to compute approximations to the torsional stiffness k_T. k_T can be determined by the following steps,

1. Find a function w defined over the cross section A, which satisfies the conditions,
   - $w = 0$ on the perimeter of the cross section A.
   - w is a function which makes the total potential energy,

     $$\Pi_{\text{total}}(w) = \int_A \frac{1}{2} \boldsymbol{\nabla} w \cdot \boldsymbol{\nabla} w \, dA - \int_A 2Gw \, dA,$$

     stationary.
2. The torsional stiffness is determined as,

   $$k_T = \int_A 2w \, dA \, .$$

The procedure of finding a good approximation to the function w in Step 1 is identical to the type of problems that have been treated in this course. This problem can be approached by employing the principle of stationary potential energy. In this procedure one follows the familiar steps:

1. Form an approximation for the function w,

   $$w(x, y) = \sum_{k=1}^{N} c_k f_k(x, y),$$

   where $f_k(x, y)$ are functions which satisfy the kinematic boundary conditions (in this case they are zero on the boundary of A) and c_k are the cofficients which must be determined. Compared to the one-dimensional case, the functions f_k depend on two variables.
2. Substitute the approximation w into the potential energy,

   $$\Pi_{\text{total}}(w) \Rightarrow \Pi_{\text{total}}(\mathbf{c}) \, .$$

3. Look for the stationary points of the potential energy,

$$\frac{\partial \Pi_{\rm t}}{\partial \mathbf{c}} \quad = \quad \mathbf{0} = \mathbf{K}\mathbf{c} - \mathbf{F}.$$

   The coefficients $\mathbf{c}$ are determined by solving the linear system of equations,

$$\mathbf{K}\mathbf{c} \quad = \quad \mathbf{F}\ .$$

4. Once the approximation is obtained,

$$w_{\rm a} \quad = \quad \sum_{k=1}^{N} c_k f_k(x, y),$$

   the approximation for the torsional stiffness is obtained as,

$$k_{T,{\rm a}} \quad = \quad \int_A 2 w_{\rm a}\, dA\ .$$

3 Exercise: Torsional stiffness of a bar with rectangular cross section

3.1 Download files

1. Download the file `torsion.zip` into your `ce130n/programs` directory and unzip it.
2. Go to the `ce130n/programs/torsion/exercise/` directory, and execute the file `init.m`. This will set the necessary paths to run the files.

YOU MUST RUN THE FILE init.m EVERYTIME YOU START UP MATLAB.

3.2 Functions used for the approximation

In this exercise you will compute approximations for the torsional stiffness k_T of bars with rectangular cross sections. The cross section has a length of $2a$ in the x-direction and a length of $2b$ in the y-direction. The shear modulus is assumed G. To apply the principle of stationary potential energy to find approximations to k_T, one must assume an approximate form for the function $w(x, y)$ which satisfies the kinematic B.C., i.e., is zero on the boundary of the rectangle. This can be achieved by assuming a cosine function in the x and y directions and taking their product. The simplest function of this form is,

$$\cos\left(\frac{\pi}{2}\frac{x}{a}\right)\cos\left(\frac{\pi}{2}\frac{y}{b}\right) \ .$$

The more general form of such a function is,

$$g_{mn}(x, y) \quad := \quad \cos\left\{(2m-1)\frac{\pi}{2}\frac{x}{a}\right\}\cos\left\{(2n-1)\frac{\pi}{2}\frac{y}{b}\right\} \ .$$

One can consider an approximate solution of the form,

$$w(x, y) \quad = \quad \sum_{m=1}^{N_x}\sum_{n=1}^{N_y} w_{mn} g_{mn}(x, y) \ ,$$

where w_{mn} are the coefficients which must be determined. Observe that a total number of $N = N_x \cdot N_y$ functions are being used for this approximation. By defining,

$$k \quad := \quad N_y(m-1) + n,$$

we can identify,

$$w_{mn} \to c_k, \qquad g_{mn}(x, y) \to f_k(x, y),$$

where this is non-other than a relabeling. By introducing this notation, one can express the approximation by the familiar single sum,

$$w(x, y) \quad = \quad \sum_{k=1}^{N} c_k f_k(x, y) \ .$$

For example, assume the case of $N_x = 2$ and $N_y = 3$. Then one has the following identification,

$$\begin{aligned} w_{11} &\leftrightarrow c_1, \\ w_{12} &\leftrightarrow c_2, \\ w_{13} &\leftrightarrow c_3, \\ w_{21} &\leftrightarrow c_4, \\ w_{22} &\leftrightarrow c_5, \\ w_{23} &\leftrightarrow c_6. \end{aligned}$$

Given k, one can determine (m,n) by the following formulas,

$$\begin{aligned} n &= [(k-1) \pmod{N_y}] + 1, \\ m &= \frac{k-n}{N_y} + 1 \,. \end{aligned}$$

Confirm that these relations hold.

The functions used for the approximation can be visualized with the following MATLAB code.

```
>> a = 1;                                % -- Length of a
>> b = 2;                                % -- Length of b
>> Nx= 2;
>> Ny= 3;
>> G = 1;                                % -- Shear modulus
>> torsion = torsion_data(a,b,Nx,Ny,G);  % -- Construct data structure
>> plotritzf(torsion);                   % -- Plots all functions
>> id= [3,5];
>> plotritzf(torsion,id);                % -- Plots only k=3,5 functions
```

Things to check:

- Make sure you understand the correspondance between k and (m,n), i.e., how you can go from one notation to the other.
- Make sure you can correlate N_x and N_y with the figures of the Ritz functions $f_k(x,y)$ shown in the plots. (e.g., The number of times the functions oscillate in each direction).

3.3 The stiffness matrix and forcing vector

Inserting the approximation into the potential energy yields $\Pi_{\text{total}}(\mathbf{c})$. The stationary condition implies,

$$\begin{aligned}
\frac{\partial \Pi_{\text{total}}}{\partial c_k} &= \int_A \frac{1}{2}\left[\frac{\partial \boldsymbol{\nabla} w}{\partial c_k}\cdot \boldsymbol{\nabla} w + \boldsymbol{\nabla} w \cdot \frac{\partial \boldsymbol{\nabla} w}{\partial c_k}\right] dA - \int_A 2G \frac{\partial w}{\partial c_k}\, dA \\
&= \int_A \boldsymbol{\nabla} w \cdot \frac{\partial \boldsymbol{\nabla} w}{\partial c_k}\, dA - \int_A 2G \frac{\partial w}{\partial c_k}\, dA = 0\,.
\end{aligned}$$

The terms involved in the calculation are,

$$\begin{aligned}
w(x,y) &= \sum_{l=1}^{N} c_l f_l(x,y)\,, \\
\frac{\partial w}{\partial c_k} &= f_k(x,y), \\
\boldsymbol{\nabla} w(x,y) &= \sum_{l=1}^{N} c_l \boldsymbol{\nabla} f_l(x,y)\,, \\
\frac{\partial \boldsymbol{\nabla} w}{\partial c_k} &= \boldsymbol{\nabla} f_k(x,y)\,.
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{\partial \Pi_{\text{total}}}{\partial c_k} &= \int_A \left[\sum_{l=1}^{N} c_l \boldsymbol{\nabla} f_l(x,y)\right]\cdot \boldsymbol{\nabla} f_k(x,y)\, dA - \int_A 2G f_k(x,y)\, dA \\
&= \sum_{l=1}^{N}\left[\int_A \boldsymbol{\nabla} f_k(x,y)\cdot \boldsymbol{\nabla} f_l(x,y)\, dA\right] c_l - \int_A 2G f_k(x,y)\, dA \\
&= \sum_{l=1}^{N} K_{kl} c_l - F_k,
\end{aligned}$$

where we have defined,

$$\begin{aligned}
K_{kl} &:= \int_A \boldsymbol{\nabla} f_k(x,y)\cdot \boldsymbol{\nabla} f_l(x,y)\, dA, \\
F_k &:= \int_A 2G f_k(x,y)\, dA\,.
\end{aligned}$$

These are the entries of the stiffness matrix $\mathbf{K}$ and forcing vector $\mathbf{F}$. For the given approximation involving cosine functions, some algebra yields the following expressions,

$$\begin{aligned}
K_{kl} &= \begin{cases} \dfrac{\pi^2 ab}{4}\left[\left(\dfrac{2m-1}{a}\right)^2 + \left(\dfrac{2n-1}{b}\right)^2\right] & (k=l), \\ 0 & (k\neq l)\,. \end{cases}, \\
F_k &= 2G\left[\frac{16ab}{\pi^2(2m-1)(2n-1)}(-1)^{m+n}\right]\,.
\end{aligned}$$

Here, (m,n) are determined from k by the relations defined previously. For example in the case of $N_x = 2$ and $N_y = 3$, to compute K_{33} for which $k=3$,

$$\begin{aligned}
n &= [(3-1) \pmod 3] + 1 = 3, \\
m &= \frac{3-3}{3} + 1 = 1\,.
\end{aligned}$$

Things to do:

- Go through the algebra and confirm that the expression for F_k given above is correct. (Write this here). (HINT: $\cos(i\pi)\cos(j\pi) = (-1)^{i+j}$).
- Complete the `for` loops in the two functions, `/core/computeK.m` and `/core/computeF.m` to compute the stiffness matrix $\mathbf{K}$ and forcing vector $\mathbf{F}$.

3.4 Computing and plotting the solution

Given the stiffness matrix $\mathbf{K}$ and forcing vector $\mathbf{F}$, one can compute the undetermined coefficients $\mathbf{c}$ to obtain the approximation,

$$w_{\mathrm{a}} = \sum_{k=1}^{N} c_k f_k(x,y) \; .$$

Things to do:

- Compute the coefficients $\mathbf{c}$ and visualize the solution with the following MATLAB code.

```
>> plotsol(torsion,c);
```

3.5 Computing the torsional stiffness

Given the approximation,

$$w_{\mathrm{a}} = \sum_{k=1}^{N} c_k f_k(x,y) \; ,$$

one can compute the torsional stiffness by,

$$\begin{aligned} k_T &= \int_A 2\left[\sum_{k=1}^{N} c_k f_k(x,y)\right] \, dA \\ &= 2\sum_{k=1}^{N} c_k \int_A f_k(x,y)\, dA \\ &= 2\sum_{k=1}^{N} c_k \frac{F_k}{2G} \\ &= \frac{1}{G}\sum_{k=1}^{N} c_k F_k \\ &= \frac{1}{G}\mathbf{c}\cdot\mathbf{F} \; . \end{aligned}$$

Things to do:

- Complete the function `/core/compute_kt.m`, which computes the torsional stiffness of a rectangular bar.
- The exact solution for the torsional stiffness can be computed with the function `/core/compute_kt_exact.m`.

```
>> kte = compute_kt_exact(torsion);
```

  Construct a plot of the relative error of k_T for the case $a = b = 1$ and $G = 1$ as N_x and N_y increase from 1 to 100. Draw this in Figure 1. (Note that this is a $\log - \log$ plot.) What is the rate of convergence? (What is the slope of the curve of this figure?).

- For a rectangular cross section with fixed area A, what is the ratio of a and b which gives the largest torsional stiffness k_T?
- Assume a rectangular cross section with the same area A as a circular cross section. Which has a larger torsional stiffness, the rectangular cross section or the circular cross section?

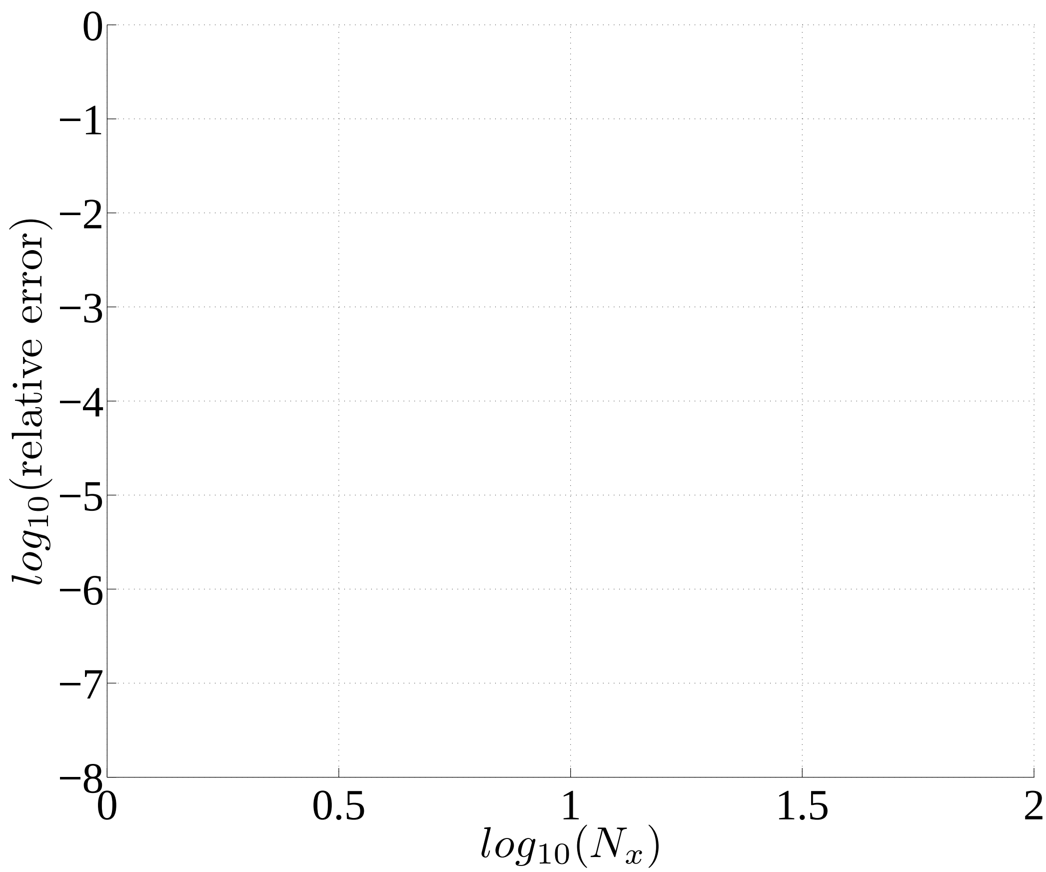

Figure 1: Log-Log plot of relative error with respect to N_x