

Mechanics of Structures (CE130N)

Lab 10

1 Objective

The objective of this lab is to understand how the principle of stationary potential energy can be applied to understand the behavior of buckling phenomenon. Before treating the continuous problem, i.e., buckling of beams, we will treat the more simple discrete system consisting of rigid elements and springs.

2 Exercise

2.1 Download files

1. Download the file `buckling.zip` into your `ce130n/programs` directory and unzip it.
2. Go to the `ce130n/programs/buckling/exercise/` directory, and execute the file `init.m`. This will set the necessary paths to run the files.

YOU MUST RUN THE FILE `init.m` EVERYTIME YOU START UP MATLAB.

2.2 Two rigid elements with by two springs

In this exercise you will investigate the buckling behavior of a system consisting of two rigid elements with two springs as shown in Figure 1. The nodes are labeled as shown. Rigid element 1 is pinned at node 1, and connected to rigid element 2 at node 2 by a rotational spring k_r [Force·Length]. There is a lateral spring k_t [Force/Length] attached to rigid element 2 at node 3. The two rigid elements have a length of L [Length]. Define the clockwise rotation of rigid element 1 as θ_1 and the clockwise rotation of rigid element 2 as θ_2 as shown in Figure 2. A vertical load of P [Force] is applied at node 3 in the direction shown.

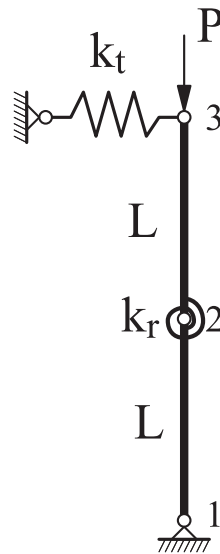


Figure 1: Setup of mechanical system

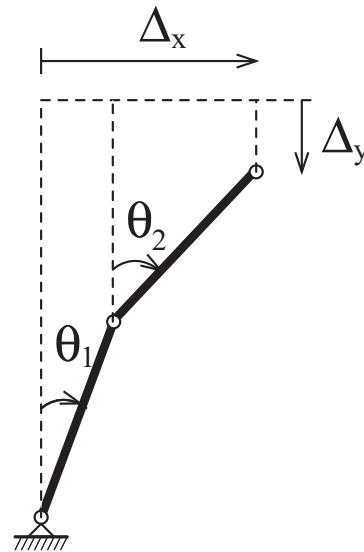


Figure 2: Deformed configuration and variables

2.2.1 Compute the total potential energy of the system

Obtain the expression for the total potential energy Π of the system as a function of θ_1 and θ_2 ,

$$\Pi(\theta_1, \theta_2) .$$

As a hint, first obtain the expression for the lateral displacement Δ_x and vertical displacement Δ_y as is shown in the Figure 2, in terms of θ_1 and θ_2 ,

$$\begin{aligned} \Delta_x(\theta_1, \theta_2), \\ \Delta_y(\theta_1, \theta_2). \end{aligned}$$

The displacement Δ_x and Δ_y is given as,

$$\begin{aligned} \Delta_x &= L \sin \theta_1 + L \sin \theta_2, \\ \Delta_y &= 2L - (L \cos \theta_1 + L \cos \theta_2) . \end{aligned}$$

The total potential energy of the system is then obtained as,

$$\begin{aligned} \Pi(\theta_1, \theta_2) &= \frac{1}{2}k_r(\theta_2 - \theta_1)^2 + \frac{1}{2}k_t\Delta_x^2 - P\Delta_y \\ &= \frac{1}{2}k_r(\theta_2 - \theta_1)^2 + \frac{1}{2}k_tL^2(\sin \theta_1 + \sin \theta_2)^2 - PL(2 - \cos \theta_1 - \cos \theta_2) . \end{aligned}$$

2.2.2 Non-dimensionalization

In many problems, it is useful to non-dimensionalize expressions to remove redundant parameters from the problem and to improve performance in numerical calculations. Here, let us non-dimensionalize the expression for the total potential energy. To proceed, one must pick representative values for the “Length” and “Force” constants. Let us make the choice,

$$\begin{aligned}cL &:= L, \\cF &:= k_t \cdot L.\end{aligned}$$

Since the potential energy Π has units of energy [Force·Length], one must divide this expression by $cF \cdot cL$. Define the non-dimensionalized version of the energy as,

$$\hat{\Pi}(\theta_1, \theta_2) := \Pi(\theta_1, \theta_2) \times \frac{1}{[cF \cdot cL]}.$$

Note that $\hat{\Pi}$ is dimensionless. Further, let us define,

$$\begin{aligned}e &:= \frac{k_r}{cF \cdot cL} && \text{Non-dimensional version of the rotational spring stiffness} \\ &&& \text{(or spring stiffness ratio)} \\ \lambda &:= \frac{P}{cF} && \text{Non-dimensional version of the load}\end{aligned}$$

Obtain the expression for $\hat{\Pi}$.

The total potential energy of the system in non-dimensional form $\hat{\Pi}$ can be rewritten as,

$$\begin{aligned}\hat{\Pi}(\theta_1, \theta_2) &= \frac{\Pi}{cF \cdot cL} \\ &= \frac{1}{2} \frac{k_r}{cF \cdot cL} (\theta_2 - \theta_1)^2 + \frac{1}{2} \frac{k_t L^2}{cF \cdot cL} (\sin \theta_1 + \sin \theta_2)^2 - \frac{PL}{cF \cdot cL} (2 - \cos \theta_1 - \cos \theta_2) \\ &= \frac{1}{2} e (\theta_2 - \theta_1)^2 + \frac{1}{2} (\sin \theta_1 + \sin \theta_2)^2 - \lambda (2 - \cos \theta_1 - \cos \theta_2).\end{aligned}$$

2.2.3 Principle of stationary potential energy for equilibrium

Apply the principle of stationary potential energy to $\hat{\Pi}$ to find the equilibrium equations which θ_1 and θ_2 must satisfy. One should obtain two non-linear equations in the variables (θ_1, θ_2) ,

$$\begin{aligned}R_1(\theta_1, \theta_2, \lambda) &= 0, \\R_2(\theta_1, \theta_2, \lambda) &= 0.\end{aligned}$$

One should consider λ as a given value, not an unknown variable.

The stationarity of $\hat{\Pi}$ implies,

$$\begin{aligned}R_1(\theta_1, \theta_2, \lambda) &:= \frac{\partial \hat{\Pi}(\theta_1, \theta_2)}{\partial \theta_1} = -e(\theta_2 - \theta_1) + (\sin \theta_1 + \sin \theta_2) \cos \theta_1 - \lambda \sin \theta_1 = 0, \\R_2(\theta_1, \theta_2, \lambda) &:= \frac{\partial \hat{\Pi}(\theta_1, \theta_2)}{\partial \theta_2} = e(\theta_2 - \theta_1) + (\sin \theta_1 + \sin \theta_2) \cos \theta_2 - \lambda \sin \theta_2 = 0.\end{aligned}$$

One can actually solve for the solution to these two non-linear equations.

$$\begin{aligned}
& \begin{cases} -e(\theta_2 - \theta_1) + (\sin \theta_1 + \sin \theta_2) \cos \theta_1 - \lambda \sin \theta_1 = 0, \\ e(\theta_2 - \theta_1) + (\sin \theta_1 + \sin \theta_2) \cos \theta_2 - \lambda \sin \theta_2 = 0 \end{cases} \\
\Leftrightarrow & \begin{cases} -e(\theta_2 - \theta_1) + (\sin \theta_1 + \sin \theta_2) \cos \theta_1 - \lambda \sin \theta_1 = 0, \\ (\cos \theta_1 + \cos \theta_2 - \lambda)(\sin \theta_1 + \sin \theta_2) = 0 \end{cases} \\
\Leftrightarrow & \begin{cases} -e(\theta_2 - \theta_1) + (\sin \theta_1 + \sin \theta_2) \cos \theta_1 - \lambda \sin \theta_1 = 0, \\ \cos \theta_1 + \cos \theta_2 - \lambda = 0 \\ \text{or} \\ \sin \theta_1 + \sin \theta_2 = 0 \end{cases} \\
\Leftrightarrow & \begin{cases} -e(\theta_2 - \theta_1) + (\sin \theta_1 + \sin \theta_2) \cos \theta_1 - \lambda \sin \theta_1 = 0, \\ \cos \theta_1 + \cos \theta_2 - \lambda = 0 \\ \text{or} \\ -e(\theta_2 - \theta_1) + (\sin \theta_1 + \sin \theta_2) \cos \theta_1 - \lambda \sin \theta_1 = 0, \\ \sin \theta_1 + \sin \theta_2 = 0 \end{cases} \\
\Leftrightarrow & \begin{cases} \sin(\theta_2 - \theta_1) = e(\theta_2 - \theta_1) \\ \lambda = \cos \theta_1 + \cos \theta_2 \\ \text{or} \\ -e(\theta_2 - \theta_1) - \lambda \sin \theta_1 = 0, \\ \theta_1 = -\theta_2 + 2n\pi \quad \text{or} \quad \theta_1 = \theta_2 + \pi + 2n\pi \end{cases} \\
\Leftrightarrow & \begin{cases} \sin(\theta_2 - \theta_1) = e(\theta_2 - \theta_1) \\ \lambda = \cos \theta_1 + \cos \theta_2 \\ \text{or} \\ -e(\theta_2 - \theta_1) - \lambda \sin \theta_1 = 0, \\ \theta_1 = \theta_2 + \pi \quad \text{or} \quad \theta_1 = \theta_2 - \pi \quad \text{since } \theta_1, \theta_2 \in [-\pi, \pi] \\ \text{or} \\ -e(\theta_2 - \theta_1) - \lambda \sin \theta_1 = 0, \\ \theta_1 = -\theta_2 \quad \text{since } \theta_1, \theta_2 \in [-\pi, \pi] \end{cases} \\
\Leftrightarrow & \begin{cases} \sin(\theta_2 - \theta_1) = e(\theta_2 - \theta_1) \\ \lambda = \cos \theta_1 + \cos \theta_2 \\ \text{or} \\ \lambda \sin \theta_1 = e\pi, \\ \theta_1 = \theta_2 + \pi \\ \text{or} \\ \lambda \sin \theta_1 = -e\pi, \\ \theta_1 = \theta_2 - \pi \\ \text{or} \\ \lambda \sin \theta_1 = 2e\theta_1, \\ \theta_1 = -\theta_2 \end{cases}
\end{aligned}$$

2.2.4 Linearization of the equilibrium equations

When one is interested in the behavior of the onset of buckling, one can look at the behavior of the system for “small” (θ_1, θ_2) near the initial equilibrium point of $(\theta_1, \theta_2) = (0, 0)$. To obtain the linearized system for this problem, one takes all the non-linear functions in the systems and constructs a Taylor series expansion around, $(\theta_1, \theta_2) = (0, 0)$.

$$\begin{aligned}\cos \theta &= 1 - \frac{1}{2}\theta^2 + \dots, \\ \sin \theta &= \theta - \frac{1}{6}\theta^3 + \dots.\end{aligned}$$

Depending on the problem, one may take a variable number of terms in the Taylor series expansion. For this problem,

$$\begin{aligned}\cos \theta &\approx 1, \\ \sin \theta &\approx \theta,\end{aligned}$$

is sufficient.

Obtain the “linearized” version of the equilibrium equations. Note that this expression should only have upto linear terms, i.e., only constants and linear functions of θ_1 and θ_2 . Write this equilibrium equation in matrix form,

$$\mathbf{K}_b \boldsymbol{\theta} = \mathbf{0},$$

where,

$$\boldsymbol{\theta} := \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix},$$

and $\mathbf{K}_b(\lambda)$ is a 2-by-2 matrix with a dependence on λ .

From the 2 non-linear equations representing equilibrium,

$$\begin{aligned}-e(\theta_2 - \theta_1) + (\sin \theta_1 + \sin \theta_2) \cos \theta_1 - \lambda \sin \theta_1 &= 0, \\ e(\theta_2 - \theta_1) + (\sin \theta_1 + \sin \theta_2) \cos \theta_2 - \lambda \sin \theta_2 &= 0,\end{aligned}$$

one can replace the non-linear functions with their linear approximations as stated in the explanation above, to obtain,

$$\begin{aligned}-e(\theta_2 - \theta_1) + (\theta_1 + \theta_2) - \lambda \theta_1 &= 0, \\ e(\theta_2 - \theta_1) + (\theta_1 + \theta_2)\theta_2 - \lambda \theta_2 &= 0,\end{aligned}$$

which can be expressed in matrix form by,

$$\left(\begin{bmatrix} 1+e & 1-e \\ 1-e & 1+e \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \mathbf{0}.$$

By defining,

$$\mathbf{K}_b(\lambda) := \begin{bmatrix} 1+e & 1-e \\ 1-e & 1+e \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

one has the desired expression of $\mathbf{K}_b \boldsymbol{\theta} = \mathbf{0}$.

2.2.5 Obtaining the buckling load

Given the equilibrium equation, $\mathbf{K}_b(\lambda)\boldsymbol{\theta} = \mathbf{0}$, one can investigate the buckling behavior. Buckling occurs when,

- the buckling load λ is a value which creates a non-trivial $\boldsymbol{\theta} \neq \mathbf{0}$ that satisfies the equilibrium equation.

A non-trivial $\boldsymbol{\theta}$ occurs only when $\mathbf{K}_b(\lambda)$ is singular. Thus in order to find the buckling load λ ,

- one can look for the λ which makes $\mathbf{K}_b(\lambda)$ singular or $\det(\mathbf{K}_b(\lambda)) = 0$.

We can split the expression for \mathbf{K}_b into two parts, the part which is independent of λ , and the part which is dependent on λ ,

$$\mathbf{K}_b(\lambda)\boldsymbol{\theta} = (\mathbf{K} - \lambda\mathbf{B})\boldsymbol{\theta} = \mathbf{0}$$

$$\Rightarrow \mathbf{K}\boldsymbol{\theta} = \lambda\mathbf{B}\boldsymbol{\theta},$$

where \mathbf{K} and \mathbf{B} are 2-by-2 matrices independent of λ . The problem of looking for combinations of a non-trivial $\boldsymbol{\theta}$ and λ which satisfy the equation $\mathbf{K}\boldsymbol{\theta} = \lambda\mathbf{B}\boldsymbol{\theta}$ is equivalent to the mathematical problem of finding the generalized eigenvalues of the matrices (\mathbf{K}, \mathbf{B}) . Note that when \mathbf{B} is equal to the identity matrix \mathbf{I} , one has the equation, $\mathbf{K}\boldsymbol{\theta} = \lambda\boldsymbol{\theta}$, which is an ordinary eigenvalue problem. Thus,

- one can find the buckling load λ and buckling mode $\boldsymbol{\theta}$ by computing the eigenvalues of the generalized eigenvalue problem, $\mathbf{K}\boldsymbol{\theta} = \lambda\mathbf{B}\boldsymbol{\theta}$.

One can compute the generalized eigenvalues of (\mathbf{K}, \mathbf{B}) in MATLAB with the command,

```
>> [V,D] = eig(K,B);
```

The columns of \mathbf{V} contain the eigenvectors and the diagonal entries of \mathbf{D} contain the eigenvalues. Thus the i th eigenvector $\mathbf{V}(:, i)$ and eigenvalue $\mathbf{D}(i, i)$ satisfy the relationship,

$$\mathbf{K}*\mathbf{V}(:, i) = \mathbf{D}(i, i)*\mathbf{B}*\mathbf{V}(:, i)$$

For the given problem,

1. Obtain the expressions for \mathbf{K} and \mathbf{B} .
2. By hand, compute the buckling load λ by solving $\det(\mathbf{K}_b(\lambda)) = 0$.
3. By hand, determine buckling mode shapes $\boldsymbol{\theta}$ for the corresponding λ , by solving $\mathbf{K}_b(\lambda)\boldsymbol{\theta} = \mathbf{0}$. Draw the mode shapes.
4. For $e = 0.5$, compute the buckling loads λ and buckling mode shapes using MATLAB. Compare them with your hand solutions.
5. Obtain the dimensional versions of the buckling load P . Recall that P has dimensions of [Force] so $P = \lambda \cdot cF$. Depending on the value of e , what is the critical buckling load P_{cr} (smallest load at which buckling occurs) and corresponding mode shape. You should observe that for some scalar value e_0 , when $e > e_0$ you will have one type of buckling mode and when $e < e_0$ you will have another type of buckling mode.

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1. Directly from the previous section on has the expressions,

$$\mathbf{K} : = \begin{bmatrix} 1+e & 1-e \\ 1-e & 1+e \end{bmatrix},$$
$$\mathbf{B} : = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

2. The buckling loads λ can be computed by solving,

$$\det(\mathbf{K}_b(\lambda)) = (1 + e - \lambda)^2 - (1 - e)^2 = 0,$$

which implies that,

$$\lambda = (1 + e) \pm (1 - e) = 2e, 2.$$

3. The buckling mode shapes for each buckling load λ can be computed as follows.

- $\lambda = \lambda_1 = 2e$:

$$\begin{aligned} \mathbf{K}_b(2e)\boldsymbol{\theta} &= \mathbf{0}, \\ \Leftrightarrow \begin{bmatrix} 1 - e & 1 - e \\ 1 - e & 1 - e \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} &= \mathbf{0} \\ \Leftrightarrow \begin{cases} \theta_2 = -\theta_1 & (e \neq 1) \\ \theta_2, \theta_1 \text{ are arbitrary} & (e = 1) \end{cases} \end{aligned}$$

Thus when $e \neq 1$ the buckling mode shape is,

$$\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \parallel \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

- $\lambda = \lambda_2 = 2$:

$$\begin{aligned} \mathbf{K}_b(2)\boldsymbol{\theta} &= \mathbf{0}, \\ \Leftrightarrow \begin{bmatrix} -1 + e & 1 - e \\ 1 - e & -1 + e \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} &= \mathbf{0} \\ \Leftrightarrow \begin{cases} \theta_2 = \theta_1 & (e \neq 1) \\ \theta_2, \theta_1 \text{ are arbitrary} & (e = 1) \end{cases} \end{aligned}$$

Thus when $e \neq 1$ the buckling mode shape is,

$$\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \parallel \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For a figure of the eigenvectors and buckling mode shapes see the next problem where they are computed for a specific case of $e = 0.5$.

4. For the case of $e = 0.5$ the buckling loads (eigenvalues) and buckling mode shapes (eigenvectors) computed from MATLAB are,

$$\begin{aligned} \lambda_1 &= 1, & \lambda_2 &= 2, \\ \mathbf{v}_1 &= \begin{bmatrix} -0.7071 \\ 0.7071 \end{bmatrix}, & \mathbf{v}_2 &= \begin{bmatrix} 0.7071 \\ 0.7071 \end{bmatrix}. \end{aligned}$$

These values match the hand solutions exactly.

5. The dimensional versions of the buckling loads are,

$$\begin{aligned} P_1 &= \lambda_1 \cdot cF = 2 \frac{k_r}{L}, \\ P_2 &= \lambda_2 \cdot cF = 2k_t L. \end{aligned}$$

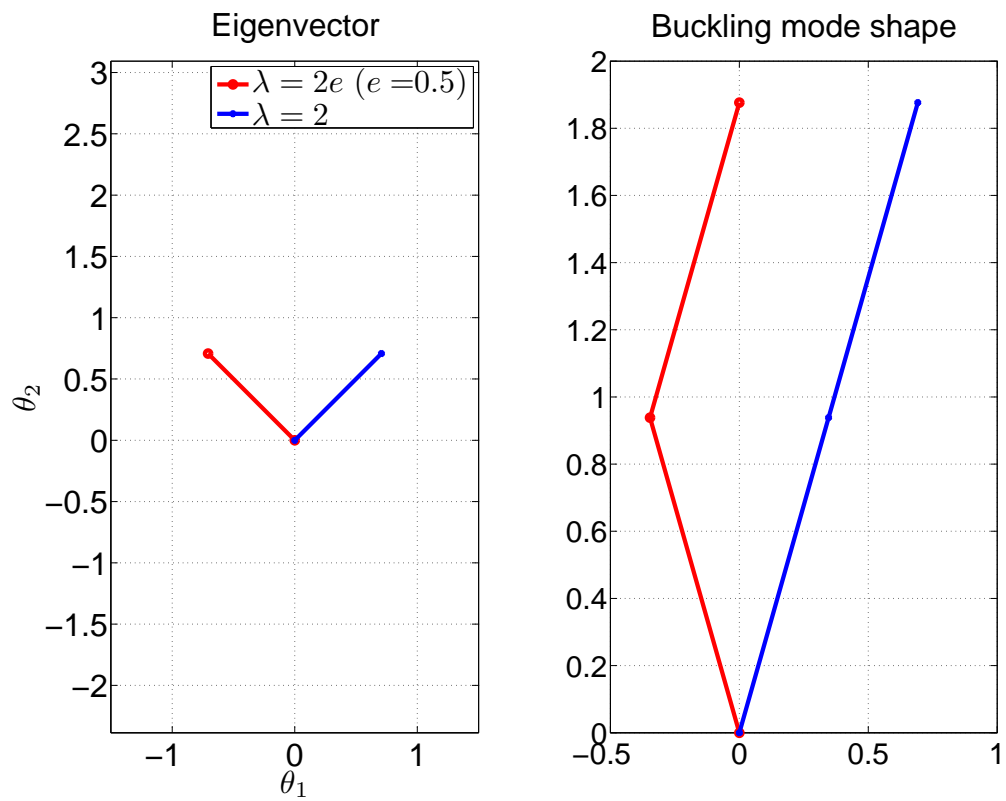


Figure 3: Eigenvectors and buckling mode shapes for buckling loads λ

When $e < 1$,

$$\lambda_{cr} = \min(\lambda_1, \lambda_2) = 2e,$$
$$P_{cr} = \lambda_{cr} \cdot cF = 2\frac{k_r}{L},$$

with corresponding buckling mode shape where the two bars move in a scissor like fashion. When $e > 1$,

$$\lambda_{cr} = \min(\lambda_1, \lambda_2) = 2,$$
$$P_{cr} = \lambda_{cr} \cdot cF = 2k_t L,$$

with corresponding buckling mode shape where the two bars move like one bar.

2.2.6 Plot the points which satisfy equilibrium

Given a load λ , one must find (θ_1, θ_2) which satisfy the two non-linear equations defining equilibrium obtained in Section 2.1.4. This can be a fairly difficult task. Here we will plot the points which satisfy equilibrium, not by directly solving the equations for a given λ but by finding the points $(\theta_1, \theta_2, \lambda)$ which satisfy,

$$\left\| \begin{bmatrix} R_1(\theta_1, \theta_2, \lambda) \\ R_2(\theta_1, \theta_2, \lambda) \end{bmatrix} \right\| \leq \text{tol},$$

where tol is some desired tolerance. Ideally we would like $\text{tol} = 0$. One can use the function `plotequilibriumv2.m` to plot the points which satisfy equilibrium through the lines of MATLAB code,

```
>> e = 1.5; % -- Spring ratio
>> param.ul_range = [-1,1]*pi; % -- Range of theta_1
>> param.u2_range = [-1,1]*pi; % -- Range of theta_2
>> param.p_range = [ 0,1]; % -- Range of lambda
>> param.ul_ndiv = 201; % -- Number of divisions of theta_1
>> param.u2_ndiv = 201; % -- Number of divisions of theta_2
>> param.p_ndiv = 201; % -- Number of divisions of lambda
>> param.eq_tol = 1e-2; % -- Tolerance mentioned above
>> plotequilibrium2v(@(u,p)eqfunc(u,p,e),param);
```

To use this function one must define a function `eqfunc.m` of the form,

```
% function [res] = eqfunc(u,p)
%
% Evaluates the residual of how much equilibrium is not
% satisfied, where u (2-by-1 vector) and p (scalar)
% is the applied load and res (2-by-1 vector)
% is a residual of how much equilibrium is not
% satisfied for the combination of u and p.
```

It is convenient to make `eqfunc.m` variable with respect to e so that one can change e without having to modify the function itself.

- How do the points which satisfy equilibrium look like in the plot. Comment on the character relating to what has been mentioned in the lecture.
- How does the plot change by varying e . At which value of e is there a fundamental change in the plots? How does the critical load P_{crit} change?
- You will see that you get two branches of solutions. Are solutions along these branches stable? Explain why or why not.

The equilibrium solutions with respect to a given load λ for the given problem for $e = 0.5$ and $e = 1.5$ are shown in Figures 4- 7. There is a fundamental change in the plot depending on whether $e < 1$ or $e > 1$. A representative case for each is shown in the given figures with $e = 0.5$ and $e = 1.5$.

- $e = 0.5$: There are two branches, the red corresponding to the scissor buckling mode shape and the blue corresponding to the single rigid bar buckling mode shape. For this case the buckling load corresponding to the scissor is smaller $\lambda_1 = 1$ and thus is the critical load. One can see in the plot that along the z axis, the red line intersects at a smaller value than the blue. Interestingly the red line branches at four locations. The crosses in

magenta denote the solutions which are stable. One observes that though the red branch is stable near the origin, it is not stable between the two branching points. The blue line is never stable. At (π, π) and $(-\pi, -\pi)$ the system exhibits behavior similar to the trivial solution branch at $(0, 0)$ and is stable for loads greater than -1.

- $e = 1.5$: There are two branches, the red corresponding to the scissor buckling mode shape and the blue corresponding to the single rigid bar buckling mode shape. For this case the buckling load corresponding to the single rigid bar mode is smaller $\lambda_2 = 2$ and thus is the critical load. One can see in the plot that along the z axis, the blue line intersects at a smaller value than the red. Compared to the case of $e = 0.5$ the red line branches at only two locations. At the origin $(0, 0)$, the red line is not stable but becomes stable after the branching points. The blue line is never stable. At (π, π) and $(-\pi, -\pi)$ the system exhibits behavior similar to the trivial solution branch at $(0, 0)$ and is stable for loads greater than -2.
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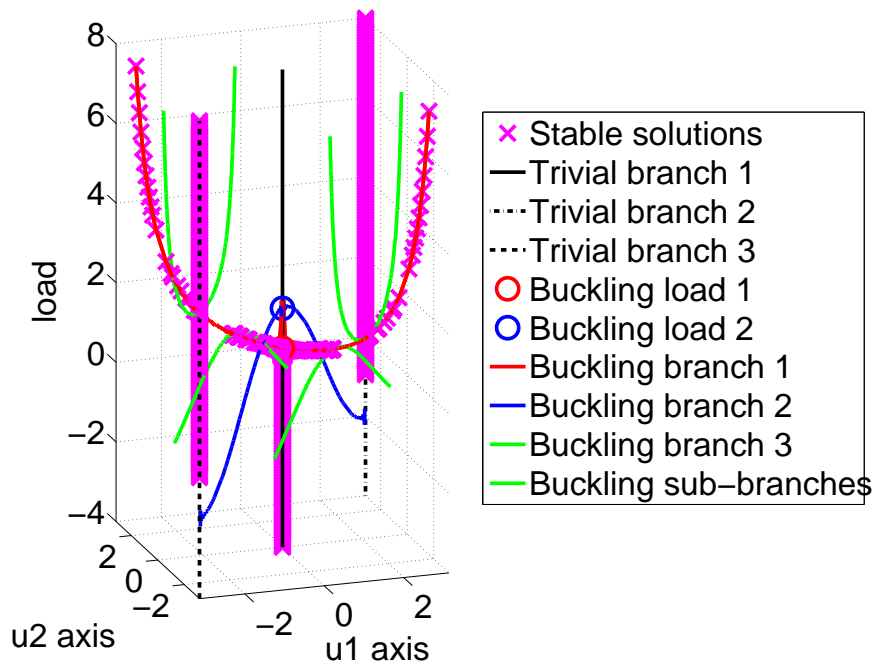


Figure 4: Equilibrium solutions, $e = 0.5$

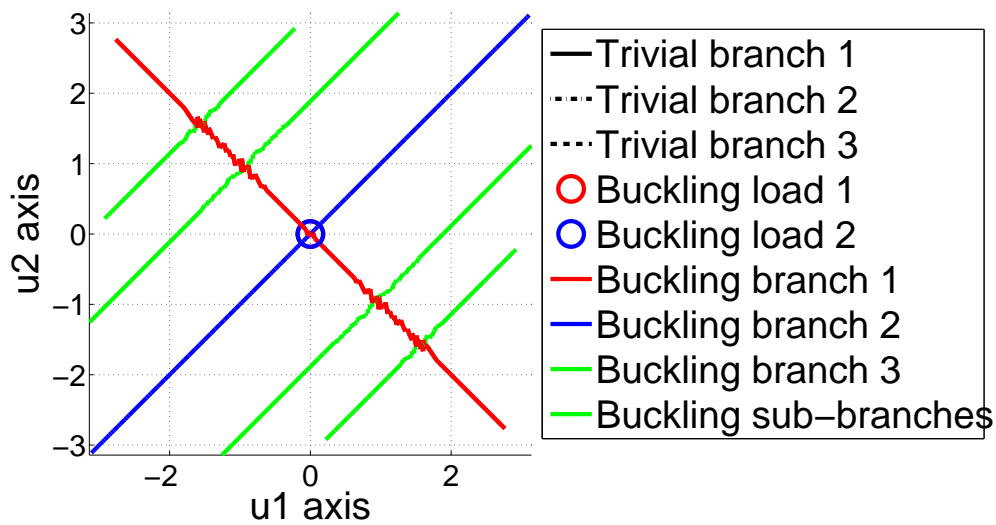


Figure 5: Equilibrium solutions, $e = 0.5$; top view

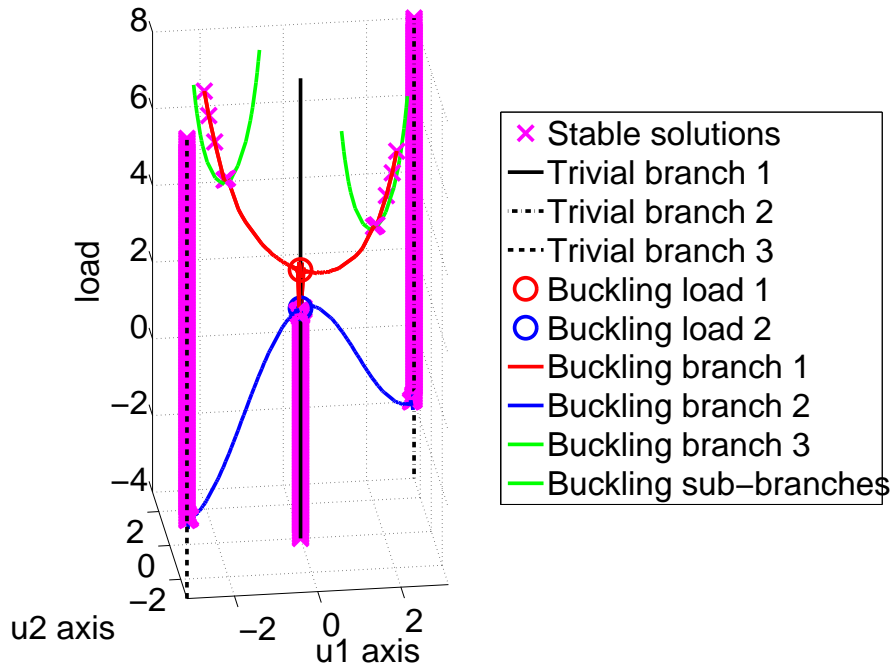


Figure 6: Equilibrium solutions, $e = 1.5$

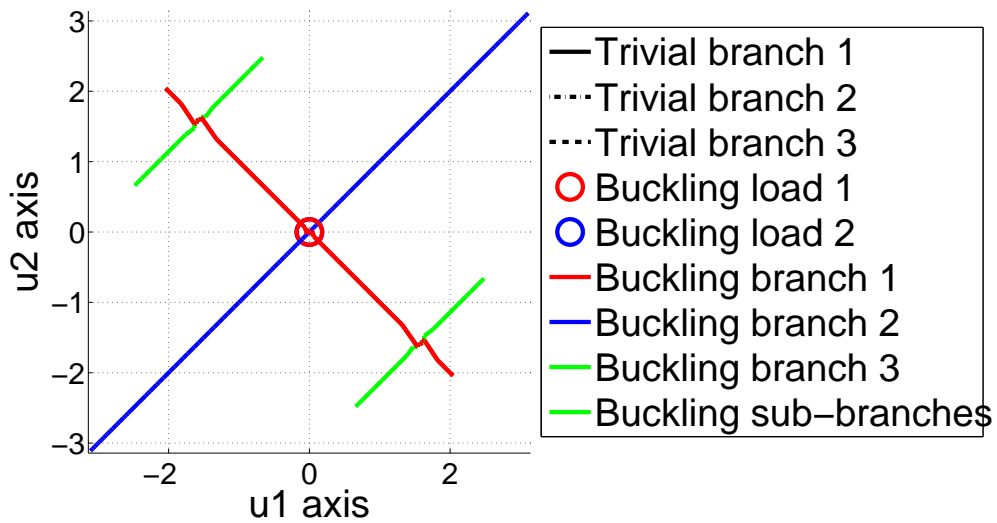


Figure 7: Equilibrium solutions, $e = 1.5$; top view

2.2.7 Plot of potential energy for varying loads

Given a load λ , one can plot the potential energy $\hat{\Pi}(\theta_1, \theta_2)$ as a function of θ_1 and θ_2 . This can be done with the function, `plotenergy2v.m`.

```
>> e = 1.5; % -- Spring ratio
>> lam = 0.0; % -- Load
>> param.u1_range = [-1,1]*pi; % -- Range of theta_1
>> param.u2_range = [-1,1]*pi; % -- Range of theta_2
>> param.u1_ndiv = 201; % -- Number of divisions in theta_1
>> param.u2_ndiv = 201; % -- Number of divisions in theta_2
>> plotenergy2v(@(u,p)efunc(u,lam,e),param);
```

To use this function one must define a function `efunc.m` of the form,

```
% function [E] = efunc(u,p)
%
% Evaluates the potential energy E (scalar) for a system
% where u (2-by-1 vector) and p (scalar)
% is the applied load.
```

It is convenient to make `efunc.m` variable with respect to e so that one can change e without having to modify the function itself.

- Take $e = 0.5$ and slowly increase λ from a value of 0. What do you observe about the energy plots and points of equilibrium. (Recall that the stationary points are the equilibrium points). Depending on the value of λ are there multiple equilibrium points? Are these equilibrium points stable?
- Take $e = 1.5$ and slowly increase λ from a value of 0. What do you observe about the energy plots and points of equilibrium. (Recall that the stationary points are the equilibrium points). Depending on the value of λ are there multiple equilibrium points? Are these equilibrium points stable?

-
- $e = 0.5$: The contour plots of the potential energy are shown for varying loads $\lambda = 0.0, 1.2, 2.2$ in Figures 8-10. These can be considered constant load λ slices of the Figure 4. The circles correspond to the stationary points of the potential energy. Through this plot, one can easily see whether a stationary point is stable or not by the arrows. The colors correspond to the branches in Figure 4.

In Figure 8 the 3 black dots are stable solutions. The blue dots are local maximum, and the green dots are saddle points.

In Figure 9 the 2 black dots at the edges are stable solutions but the black dot in the middle is a saddle point and thus not a stable solution. The two red dots are stable solutions. The two blue dots are local maximum and not stable solutions.

In Figure 10 the 2 black dots at the edges are stable solutions but the black dot in the middle is a local maximum and thus not a stable solution. The two red dots are stable solutions. The four green dots are saddle points and not stable solutions.

- $e = 1.5$: The contour plots of the potential energy are shown for varying loads $\lambda = 0.0, 3.5, 7.0$ in Figures 11-13. These can be considered constant load λ slices of the Figure 6. The circles correspond to the stationary points of the potential energy. Through this plot, one can easily see whether a stationary point is stable or not by the arrows. The colors correspond to the branches in Figure 6.

One can make similar arguments about the stationary points as in the case of $e = 0.5$.

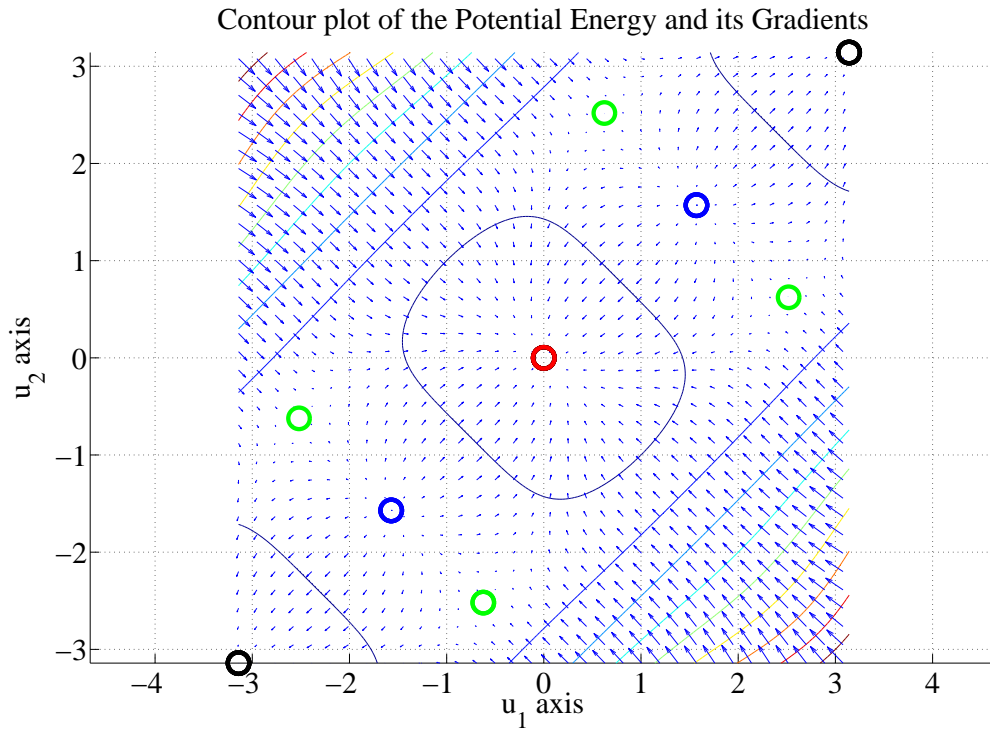


Figure 8: Equilibrium solutions, $e = 0.5, \lambda = 0.0$

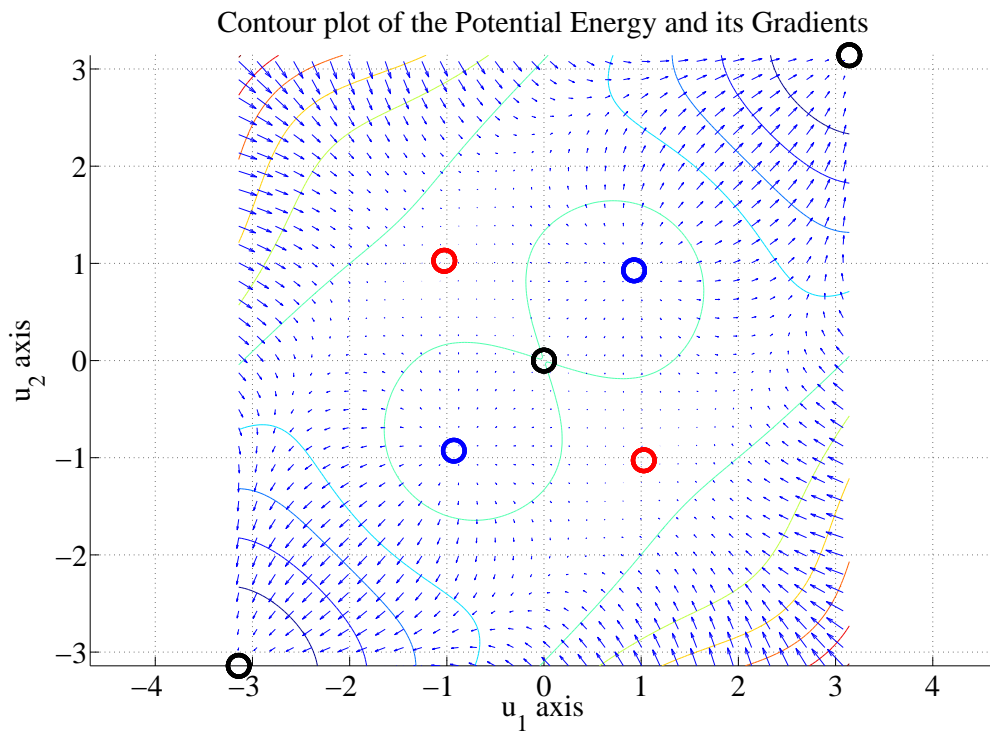


Figure 9: Equilibrium solutions, $e = 0.5, \lambda = 1.2$

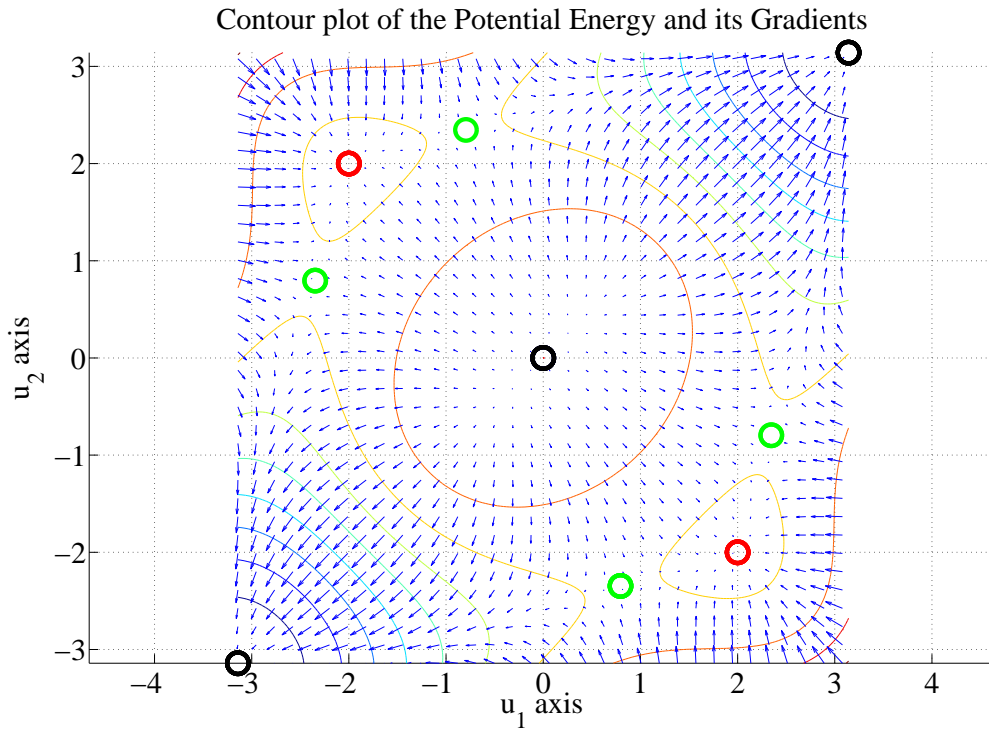


Figure 10: Equilibrium solutions, $e = 0.5, \lambda = 2.2$

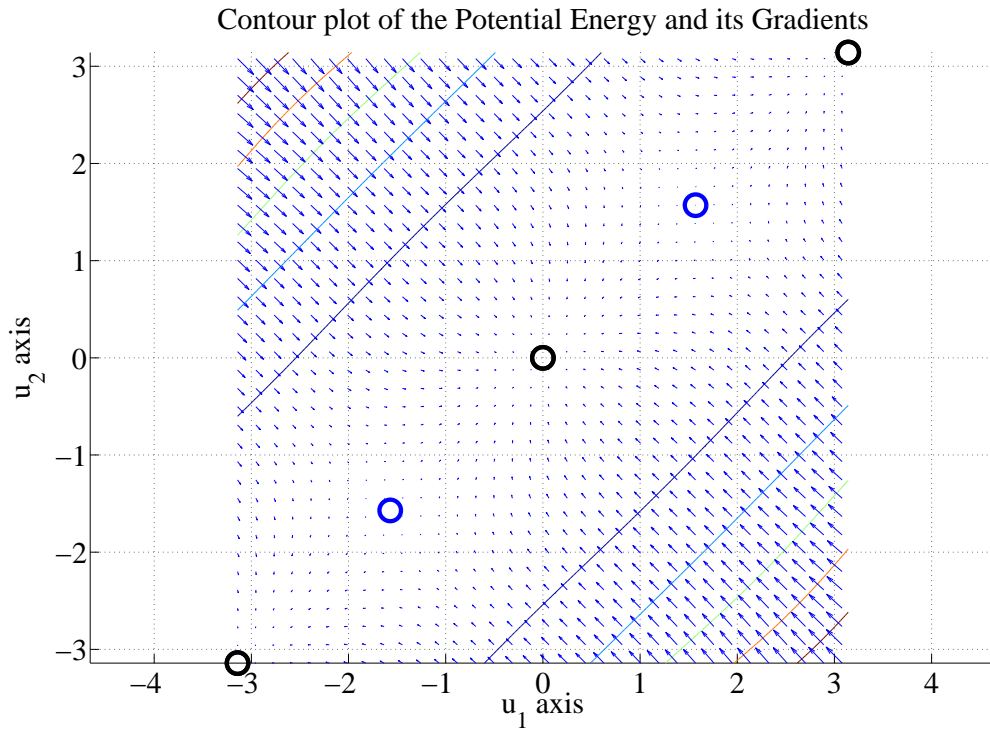


Figure 11: Equilibrium solutions, $e = 1.5, \lambda = 0.0$

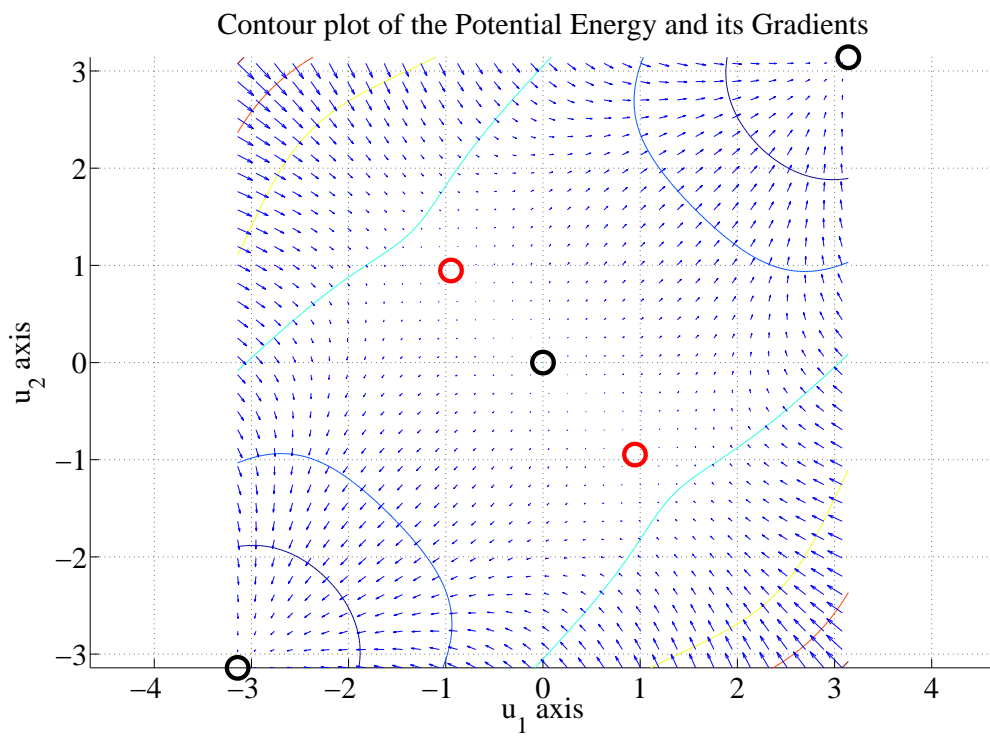


Figure 12: Equilibrium solutions, $e = 1.5, \lambda = 3.5$

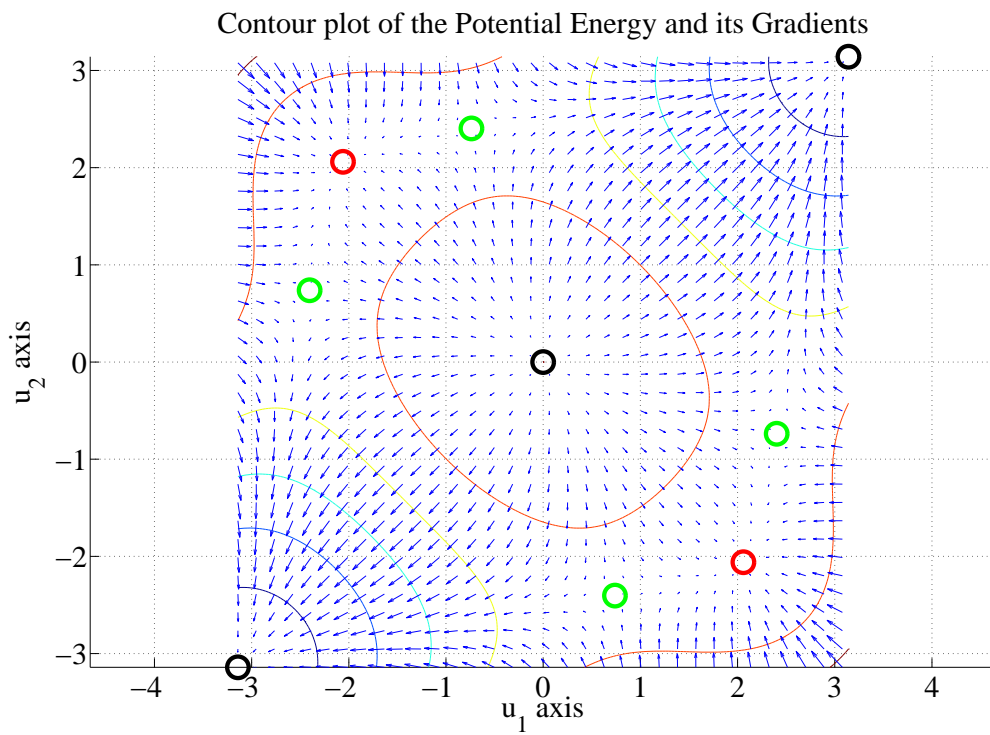


Figure 13: Equilibrium solutions, $e = 1.5, \lambda = 7.0$