c/

Henky → Hencky everywhere

For the remainder of this introductory chapter we will restate some basic concepts from elementary physics that we will require throughout the reminder of the book. In particular, we will review some concepts of force systems and equilibrium. The reader who is already comfortable with such notions, can skip directly to Chapter 2 without any loss.

1.1 Force systems

Forces are the agents that cause changes in a system. In this book we are concerned with the deformation state of solids and thus we will be concerned with traditional forces – those that cause motion of a system. In general, forces are abstract quantities that cannot be directly measured. Their existence is confirmed only via the changes they produce. Notwithstanding, all of us have an intuitive feeling for forces, and we will take advantage of this and not dwell any further upon their philosophical aspects.

In order to move or deform a body or system one needs to apply forces to it. In this regard there are two basic ways in which one can apply a force:

(1) on the surface of a body with a surface traction, a force per unit area, or

(2) throughout the volume of a body with a body force, a force per unit volume.

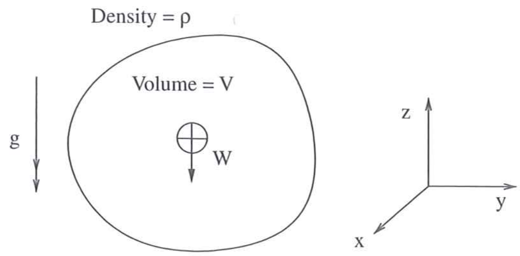
Common examples of surface tractions would be, for example, drag forces on a vehicle, the pressure between your feet and the ground when you walk, or the forces between flowing water and turbine blades in a generator in a hydroelectric dam. Examples of body forces include gravitational forces and magnetic forces.

A fundamental postulate of mechanics further states that for every force system acting on a body there is an equal and opposite force system acting on the ~~body~~ which cause~~s~~ the forces. In various forms, this reaction force principle is known as Newton's Third Law.

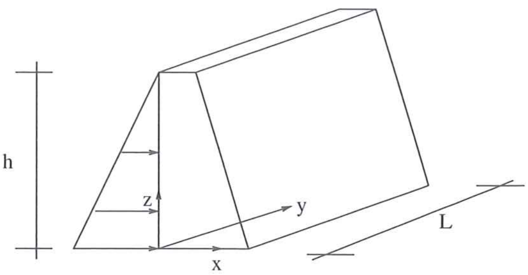
bodies

1.1.1 Units

The units of force in the SI system (Le Système International d'Unités) are Newtons (N). This is a derived unit from those of mass, length, and time. 1 Newton is equal to 1 kilogram times 1 meter per second squared: 1 N $=$ 1 kg 1 m/s^2. In the USCS (United States Customary System) the units of force are pounds-force (or simply pounds) and are denoted by the abbreviations lb or lbf. In this system 1 pound-force is exactly equal to 1 pound-mass times 32.1740 feet per second squared: 1 lbf $=$ 1 lbm 32.1740 ft/s^2. In the USCS the pound-mass (lbm) is defined exactly in terms of the kilogram – 0.45359237 kg $=$ 1 lbm. When using the USCS one should exercise some caution, due to the occasionally used unit of slug for mass. To accelerate a mass of 1 slug



(a) Homogeneous body under the influence of gravity - a distributed body force system.



(b) Dam under the influence of a distributed surface force system.

Fig. 1.1 Systems with distributed loads.

math bold for the e

$\underset{\sim}{e}_x$

the traction is a function of position on the dam face. Local to each point is a different amount of force, and these amounts need to be summed. The appropriate mathematical device in this regard is integration, and the total resultant force will be $\boldsymbol{R} = \int_A \boldsymbol{t}(y,z)\,dydz = (1/2)h^2 L\rho g$. One way of thinking about the total resultant force is that it represents the (spatial) average of the force system times the area over which it acts. The characterization of a force system solely by its resultant force is quite useful, but in many situations too crude for effective analysis. One important refining concept is that of the first moment of a force distribution or simply moment. Moments of functions are defined relative to reference points which can be freely chosen. If we take as our reference point a point labeled $\boldsymbol{x}_o$, then an effective definition of the (first) moment of a surface force system about this point is given by $\boldsymbol{M}_o = \int_A (\boldsymbol{x} - \boldsymbol{x}_o) \times \boldsymbol{t}(\boldsymbol{x})\,dA$. This moment gives additional information about the spatial distribution of the force system.

Remarks:

(1) Taken together, $\{\boldsymbol{R}, \boldsymbol{M}_o\}$ give us two parameters to characterize a force system – at least approximately.

(2) If the force system involves body forces, then the appropriate definitions are $\boldsymbol{R} = \int_V \boldsymbol{b}(\boldsymbol{x})\,dV$ and $\boldsymbol{M}_o = \int_V (\boldsymbol{x} - \boldsymbol{x}_o) \times \boldsymbol{b}(\boldsymbol{x})\,dV$.

(3) Knowledge of $\{\boldsymbol{R}, \boldsymbol{M}_o\}$ is sufficient to fully characterize the effects of a force system acting on a rigid body.

(4) Force systems where $\boldsymbol{M}_o$ is zero are called single force systems or point force systems. Force systems where $\boldsymbol{R}$ is zero are called force couple systems. If both $\boldsymbol{R}$ and $\boldsymbol{M}_o$ are zero, then the force system is said to be self-equilibrated or in equilibrium.

1.2.2 Equivalent forces systems

The characterization of a force system in terms of a resultant force and a moment is dependent upon the choice of a reference point. Since the reference point is arbitrary, the representation is not unique. If we choose another reference point $\boldsymbol{x}_p = \boldsymbol{x}_o + \boldsymbol{a}$, where $\boldsymbol{a}$ is the vector from $\boldsymbol{x}_o$ to the new reference point $\boldsymbol{x}_p$, then it is easy to see that the new characterization of our force system is $\{\boldsymbol{R}, \boldsymbol{M}_p\}$, where $\boldsymbol{M}_p = \boldsymbol{M}_o - \boldsymbol{a} \times \boldsymbol{R}$. This new characterization is considered equivalent to the first.

Remarks:

(1) Note that if the force resultant, $\boldsymbol{R}$, is zero, then the first moment of the force system is independent of the reference point. In particular, this tells us that a force couple system will always be a force couple system~~s~~ regardless of reference point.

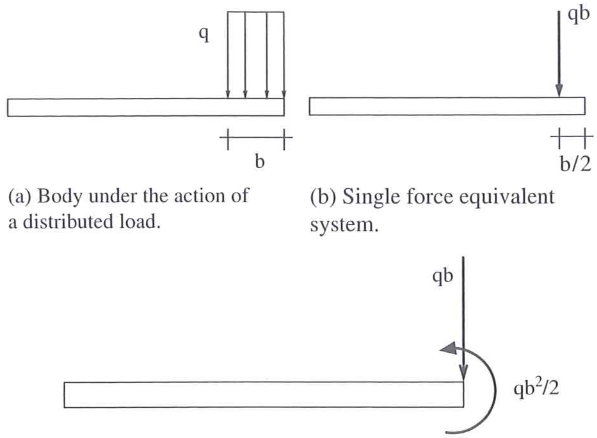
(2) A force system which is in equilibrium will always be independent of the reference point. in equilibrium

(3) All the points along the line $\boldsymbol{x}(\mu) = \boldsymbol{x}_o + \mu \boldsymbol{n}_R$ have the same moment resultant characterization, where $\mu \in \mathbb{R}$ and $\boldsymbol{n}_R = \boldsymbol{R}/\|\boldsymbol{R}\|$; this is the locus of points through $\boldsymbol{x}_o$ in the direction of $\boldsymbol{R}$.

(4) The notion of equivalent must be carefully understood. The nomenclature stems from the study of rigid body mechanics. In the framework of rigid bodies, equivalent force systems have the exact same effect on a given rigid body. In a deformable body this is not true. What is true, however, is that if carefully chosen and interpreted, equivalent force systems will have almost the same effect on a given deformable body. We will see this more clearly later in the book.

Example 1.1

Equivalent forces system~~s~~. As an example, consider the bar shown in Fig. 1.2(a). The bar is loaded with a constant distributed load. In Fig. 1.2(b) one possible characterization of the force system is shown. It consists of a single force acting at a point a distance $b/2$ from the end of the bar. In Fig. 1.2(c) a second equivalent characterization of the force system is shown. All three forces systems are equivalent according to our definition of equivalence. If the bar is rigid, then all three force systems will have the exact same effect on the behavior of the bar. If, however, the bar is deformable then the three systems will all affect the bar in



(a) Body under the action of a distributed load.

(b) Single force equivalent system.

(c) Second characterization of the force system.

Fig. 1.2 Three equivalent force systems.

different ways. This difference will manifest itself primarily in the region of length b from the right-hand end of the bar. At distances greater than b from the end of the bar, the effect of the three loading systems will be nearly identical.

Remarks:

(1) The load shown in Fig. 1.2(a) is an example of a parallel distributed loading system. That is, at every point where the load acts, the load points in the same (constant) direction in space. For such loading systems, one can always find a single force characterization of the loading (as shown in Fig. 1.2(b)). The point where this single force acts is the centroid of the loading. For example, if the distributed force is given as $\boldsymbol{t} = t\boldsymbol{e}$ over an area A where $\boldsymbol{e}$ is a constant vector, then the single force equivalent load acts at ~~the~~ point $\boldsymbol{x}_R = \int_A t\boldsymbol{x}\,dA / \int_A t\,dA$. In the case where t is also a constant, then $\boldsymbol{x}_R$ coincides with the geometric centroid of the area A, $\boldsymbol{x}_c = (1/A)\int_A \boldsymbol{x}\,dA$.

math bold e

↓

$\mu\, e_t$ any

, where $\mu \in \mathbb{R}$

and one takes $\mu = 0$

1.3 Work and power

The effect of forces is often characterized by the scalar concepts of work and power. The power of a force is defined as the scalar product of the force and the velocity of the material point where it acts: $\mathcal{P} = \boldsymbol{F}\cdot\boldsymbol{v}$. For distributed loads, the power is defined by $\mathcal{P} = \int_A \boldsymbol{t}\cdot\boldsymbol{v}\,dA$ and $\mathcal{P} = \int_V \boldsymbol{b}\cdot\boldsymbol{v}\,dV$. The work of a force system is defined as the time integral of the power over the time interval of application of the force system: $W = \int_I \mathcal{P}(t)\,dt$, where I is some interval of time.

a/

energy, then it does not effect the force or the work expression – as the constant always drops out.

1.3.2 Conservative systems

In mechanics one also has the concept of conservative systems. Conservative systems are those that conserve their total energy; i.e. they do not dissipate energy. A common example of a conservative system is a gravitational pendulum. In such a system, the energy changes continuously from kinetic energy to potential energy of the mass of the pendulum, but at all times the sum of the two is constant. Later on, we will deal with deformable bodies under the action of various conservative loading systems. When we assume the bodies to be elastic (without dissipation) then we will be able to exploit the notion of energy conservation as long as we consider our system to be the deformable body plus the loading system.

1.4 Static equilibrium

in some inertial frame/

Static equilibrium of a material system is a property of the material system and the force system acting upon it. A system is in a state of static equilibrium if all material points in the system have zero velocity and remain so as long as the force system acting on the material system does not change. Determining the conditions required for a static equilibrium is an important part of the analysis of many engineering systems, as many systems are designed to be in a state of equilibrium. To actually determine working relations, we will need to make a hypothesis about equilibrium.

1.4.1 Equilibrium of a body

The central axiom of mechanics states:

For a body to be in static equilibrium the resultant force and moment acting on every subset of the body must be equal to zero.

Thus, if we have a body Ω, then for every subset $B \subset \Omega$ with a force system acting on B which is characterized by $\{\boldsymbol{R}, \boldsymbol{M}_o\}_B$, we must have $\{\boldsymbol{R}, \boldsymbol{M}_o\}_B = \{\boldsymbol{0}, \boldsymbol{0}\}$. Another way of saying the same thing is that the resultant force system acting on any part of a body must be in equilibrium for the whole body to be in static equilibrium. This axiom takes as its origin Newton's Second Law.

1.4.2 Virtual work and virtual power

The concept of static equilibrium is very intuitive – the system in question is unchanging, i.e. static. The determination of static equilibrium

Solution
Starting from eqn (2.77) we have

$$\frac{1}{2}P\Delta = \int_V \frac{1}{2}\sigma\varepsilon\,\mathrm{d}V \tag{2.78}$$

$$= \int_0^L \int_A \frac{1}{2}\sigma\varepsilon\,\mathrm{d}A\,\mathrm{d}x \tag{2.79}$$

$$= \int_0^L \frac{1}{2}\sigma\varepsilon A\,\mathrm{d}x \tag{2.80}$$

$$= \int_0^L \frac{1}{2}\left(\frac{P}{A}\right)\left(\frac{P}{AE}\right) A\,\mathrm{d}x \tag{2.81}$$

$$= \frac{1}{2}\frac{P^2L}{AE} \tag{2.82}$$

If we now cancel $\frac{1}{2}P$ from both sides we find $\Delta = PL/AE$ – a result we had from before. Thus we see that by using conservation of energy it is possible to determine deflections in elastic systems. To see the real power of this method, consider the next example.

Example 2.13

Deflection of a two-bar truss. Consider the two-bar truss shown in Fig. 2.23. Find the horizontal deflection of the truss at the point of application of the load.

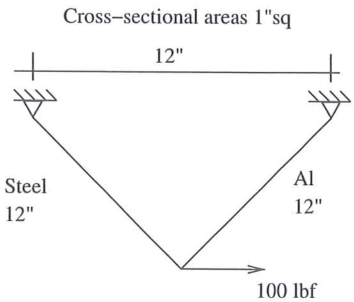


Fig. 2.23 Two-bar truss.

Solution
Using energy conservation we have that

$$W_{\text{stored}} = W_{\text{Al}} + W_{\text{Steel}}$$
$$= \left(\frac{P^2L}{2AE}\right)_{\text{Al}} + \left(\frac{P^2L}{2AE}\right)_{\text{Steel}}. \tag{2.83}$$

From statics one has that $P_{\text{Steel}} = 100$ and $P_{\text{Al}} = -100$. Thus,

$$W_{\text{stored}} = 100^2\left[\left(\frac{L}{2AE}\right)_{\text{Al}} + \left(\frac{L}{2AE}\right)_{\text{Steel}}\right]. \tag{2.84}$$

lbf b 2x

Setting this equal to the work done on the truss ($W_{\text{in}} = \frac{1}{2}100\Delta_H$) gives the final result:

$$\Delta_H = 100\left[\left(\frac{L}{AE}\right)_{\text{Al}} + \left(\frac{L}{AE}\right)_{\text{Steel}}\right] = 16 \times 10^{-5} \text{ inches.} \tag{2.85}$$

Remarks:

(1) Note that the method only gives the deflection in the direction of the applied load. No information is garnered about the vertical

deflection. Nonetheless, it is easy to see how this simple idea makes a somewhat complex problem rather tractable.

Example 2.14

Elastic impact barrier. Consider a rigid object of weight W dropping from a height h above an elastic bar of length L, area A, and modulus E as shown in Fig. 2.24. Find the maximum force carried by the elastic bar before elastic release.

Solution

This system is conservative. The total initial energy of the system is $W(h+L)$. After the object drops it impacts the bar and deforms it. The maximum force will occur at maximum deformation, Δ, which we take as positive in contraction for this problem. At the point of maximum deformation, the weight is momentarily at rest and has zero kinetic energy. At this state, the total energy of the system will be given by $W(L-\Delta)+\frac{1}{2}\frac{P^2L}{AE}$ – the potential energy of the weight plus the stored energy in the elastic bar. By conservation of energy we have that

>0

$$W(h+L) = W(L-\Delta) + \frac{1}{2}\frac{P^2L}{AE}. \tag{2.86}$$

Noting that $\Delta = PL/AE$, we find through a little algebra that

where $P>0$ in compession here,

$$P^2 - 2WP - 2WhAE/L = 0. \tag{2.87}$$

Solving shows

$$P = W[1 \pm \sqrt{1+2h/\Delta_s}], \tag{2.88}$$

where $\Delta_s = WL/AE$ is the static deflection of the bar.

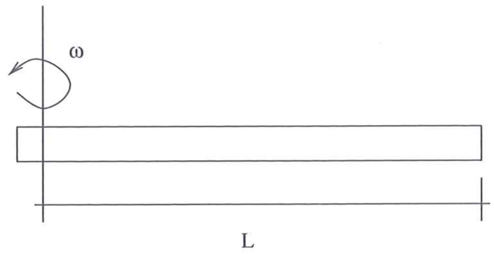
which implies a tensile force

>0

There are two solutions to this problem, but one is physically meaningless – the one giving negative values of P. Note also that in this problem independent of h and Δ_s the minimum force in the bar before elastic release is $2W$.

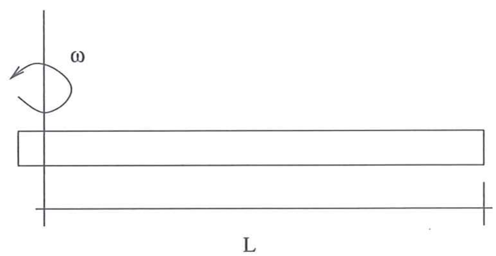
2.6 Stress-based design

The methods we have developed in this chapter can be easily incorporated into a stress-based design methodology for structural members that carry their loads axially (such as truss bars). Simply put, we can take our system of interest and apply our methodology to determine the stress field in the bar. Such information can then be used in simple stress-based design procedures when the allowable stresses for the materials of the body are known. The allowable stresses σ_a are normally either the yield stress of the material σ_Y or the ultimate stress of the material



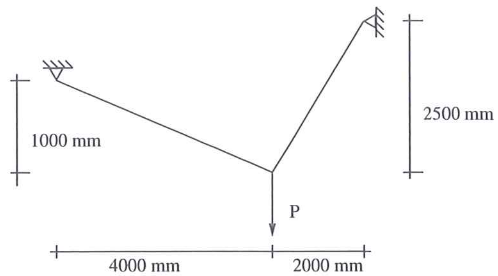
Let the length be given as sL_o, where s is a scale factor and $L_o = 1$ m. Let the cross-sectional area be given as $s^2 A_o$ where $A_o = 25$ cm^2. Thus when $s = 1$, the scale of the rod is quite macroscopic (1 m long with a 20 to 1 aspect ratio). When $s = 10^{-6}$, we will be looking at a rod with nanoscale dimensions but the same aspect ratio. Plot ω_c versus s for $s \in (10^{-6}, 1)$. Assume the material to be copper: $E = 128$ GPa, $\sigma_{\max} = 100$ MPa, $\rho = 8960$ kg/m^3. Your plot should have properly labeled axes etc.

(2.31) Consider a bar of length L with constant EA and constant density ρ. The bar is supported by a fixed pivot and spun about it at angular frequency ω. Doing so produces a distributed body force $b(x) = A\rho\omega^2 x$, where x is measured from the pivot. Find the maximum and minimum strains and their locations.



(2.32) For the system described in Exercise 2.31, find an expression for the maximum displacement and its location.

(2.33) For the two-bar truss shown, find an expression for the vertical deflection at the point of application of the load. Assume both truss bars are made from the same material with modulus E and have constant cross-sectional areas A.



(2.34) How much energy does it take to compress a round bar of diameter 5 mm and length 300 mm to a length of 299.9 mm. Assume $E = 200$ GPa. Assume the load is applied at the ends of the bar with two opposing forces.

(2.35) Using an energy method verify the solution to Exercise 2.19. First, using your solution to Exercise 2.19, show that the work-in expression is given as $\frac{1}{n+1} P\Delta$, where n is the material exponent, P is the total applied load, and Δ is the total elongation. Second, equate the work-in to the work-stored. Now, using the notion of energy conservation, verify your elongation formula for Exercise 2.19.

elastic

$$\sigma = \frac{F}{A_o}\sin^2(\theta) \tag{3.7}$$

and for the average shear stress

$$\tau = -\frac{F}{A_o}\cos(\theta)\sin(\theta). \tag{3.8}$$

It should be clear that our values of average normal and shear stress strongly depend on the section cut we make, even though the state of load in the material is fixed.

This brings up the natural question: For which section cut angles are the stresses ~~maximum~~? From a plot (see Exercise 3.2), one finds that the average shear stress will be extremal at an angle of $\theta = \pm\pi/4$ with a ~~value of τ~~ $= F/2A_o$, and the average normal stress will be maximum at an angle of $\theta = \pi/2$ with a value of $\sigma = F/A_o$. The importance of these extremal values is that different materials are more sensitive to different kinds of stress. Loosely, one can can say that when designing with brittle materials such as cast iron and concrete one has to worry about tensile normal stresses, but when designing with ductile materials such as mild steel and aluminum one has to worry about shear stresses.

extremal

magnitude of $|\tau|$

3.1.2 Design with average stresses

At this stage one can already perform some very basic engineering analysis of the stresses in a body. If the body is statically determinate we can define a simple procedure for stress analysis that is an extension of statics:

(1) Use equilibrium (statics) to find support reactions.
(2) Determine the internal forces[2] in the system.
(3) Resolve the internal forces into normal and tangential components.
(4) Compute the average normal and average shear stresses.

[2] Note that we have intentionally not mentioned internal moments here; to discuss the connection between internal moments and stress will require us to refine our definition of stress one more time.

Example 3.1

Analysis of a shear key. Figure 3.3 shows a gear on a drive shaft. In these types of system the keys which couple the gears to the shaft are designed as the weakest link. In this way, failure due to overloading destroys an inexpensive part such as the key instead of an expensive part such as a shaft or gear. Given the dimensions of the system and knowledge of the yield stress (in shear) for the key, find an expression for the maximum design torque when allowing for a safety factor on yield.

Solution
Start by isolating the shaft with half of the shear keys left intact. Summing the moments about the shaft center gives $2FR = M$; see Fig. 3.3 (bottom left). The force acts parallel to a surface with area Lt; thus the shear stress in the key is

$$\sigma = \frac{\Delta F_n}{\Delta A} \tag{3.17}$$

$$\tau = \frac{\Delta F_t}{\Delta A}. \tag{3.18}$$

Here n refers to the outward normal to the patch and t to some direction tangential to the patch.

In the limit as $\Delta A \to 0$ the patches will shrink to points, and we will recover a definition of stress at each point on the cross-section.

3.2.1 Nomenclature

At each point in a body, one can consider an infinite number of section cuts which pass through a particular point. This will lead to an infinite number of possible normal stresses and shear stresses at a point. Remarkably, knowing the normal stress and the shear stress on just three orthogonal planes passing through the point of interest permits one to know the stresses on any other section cut through the given point. In two-dimensional problems this is the case with only two orthogonal planes. In this regard, when stating the stress at a point there is a standard convention that is employed in choosing these orthogonal planes. We adorn our symbol σ for stress with two subscripts:

$$\sigma_{\sqcup\sqcup}. \tag{3.19}$$

The first subscript is used to define the section cut orientation, and takes on the values $\{x, y, z\}$ or $\{1, 2, 3\}$. An 'x' is used, for instance, if the section cut normal is in the x-direction. The second subscript is used to indicate the component of the force vector involved, and takes on values $\{x, y, z\}$ or $\{1, 2, 3\}$.

For example, the stress component σ_{xx} corresponds to a normal stress on a section cut with normal in the x-direction. The symbol σ_{12}, for instance, corresponds to a shear stress in the 2-direction on a section cut with normal in the 1-direction. Quite often one will see the symbol τ used instead of σ when the subscripts are not equal, and when the subscripts are equal one will often see the second subscript dropped. The convention for reporting the stress components at a point is to place them in a matrix using the following ordering:

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}. \tag{3.20}$$

Just as for forces (i.e. vectors) there is also a pictorial representation for stresses (tensors). The convention is to draw a set of orthogonal planes with arrows in the positive directions and to place the magnitude of the various components by the arrows. This is illustrated in Fig. 3.7, where the planes correspond to the section cuts and the arrows correspond to the directions of the force components.

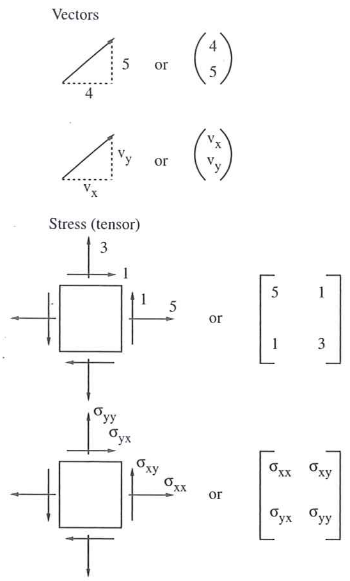


Fig. 3.7 Vector and tensor drawing conventions.

in the x-direction. The stresses that will contribute to forces in the x directions will be σ_{xx} and σ_{yx}. The x-component of the body force, b_x, will also contribute to the forces in the x-direction. If we assume that the stresses are relatively constant over the sides then we can add up the total force in the x direction very easily. To do this, on the bottom and left sides we will assume the stresses to have the values of the stresses at the point (x, y). On the right and top the stresses will have different values that we will define in terms of their change from those on the bottom and left. Multiplying the stresses by the areas over which they act gives the appropriate force values, and multiplying the body force by the volume over which it is distributed gives a force. Summing all the forces results in

$$\begin{aligned}\sum F_x = &(\sigma_{xx} + \Delta\sigma_{xx})t\Delta y - (\sigma_{xx})t\Delta y \\ &+ (\sigma_{yx} + \Delta\sigma_{yx})t\Delta x - (\sigma_{yx})t\Delta x \\ &+ b_x t\Delta x\Delta y = 0.\end{aligned} \tag{3.32}$$

In the last expression, t represents the thickness of the two-dimensional body. It is only used to get the units correct; its exact value is not required, since it drops out of the equations. Now, divide through by $t\Delta x\Delta y$ to give

$$\frac{\Delta\sigma_{xx}}{\Delta x} + \frac{\Delta\sigma_{yx}}{\Delta y} + b_x = 0. \tag{3.33}$$

Taking the limit as $\Delta x, \Delta y \to 0$ gives the partial differential equation

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{yx}}{\partial y} + b_x = 0. \tag{3.34}$$

Thus the statement "sum of the forces in the x-direction equals zero" is replaced by a partial differential equation. We can follow the same argument for the y direction to obtain the relation:

$$\frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + b_y = 0. \tag{3.35}$$

For moment equilibrium we will take moments about the lower left corner. Figure 3.11 shows all the stresses components that will contribute to the moment. Note that it suffices to only include the shear stresses.[3] Computing the moment about the lower left corner then gives:

$$\begin{aligned}\sum M_z = &(\sigma_{xy} + \Delta\sigma_{xy})t\Delta y\Delta x - (\sigma_{yx} + \Delta\sigma_{yx})t\Delta x\Delta y \\ &+ b_x t\Delta x\Delta y(\Delta y/2) - b_y t\Delta x\Delta y(\Delta x/2) = 0.\end{aligned} \tag{3.36}$$

Now divide by $t\Delta x\Delta y$ and take the limit as $\Delta x, \Delta y \to 0$. This yields the result:

$$\sigma_{xy} = \sigma_{yx}. \tag{3.37}$$

[3] If normal stresses are included, then eqn (3.36) will pick up two additional terms – one proportional to $\Delta\sigma_{yy}\Delta x^2 t$ and one proportional to $\Delta\sigma_{xx}\Delta y^2 t$. Upon division by $t\Delta x\Delta y$, these terms will go to zero. For a more rigorous derivation of eqn (3.37), see eqns (3.46) – (3.49).

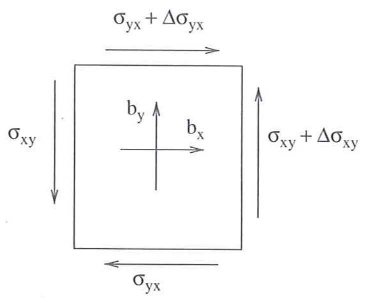


Fig. 3.11 Small piece of material with stresses and body forces which contribute to moment about the z axis.

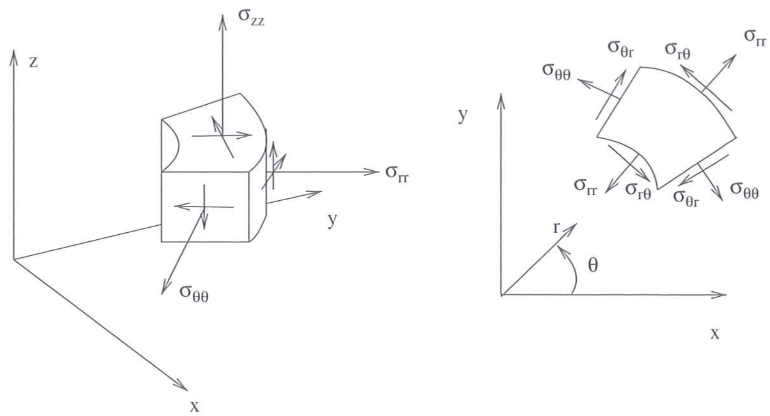


Fig. 3.14 Cylindrical stresses. (Right) Three-dimensional representation with some stresses labeled. (Left) Two-dimensional case with all stresses labeled.

and for spherical coordinates one has

$$\begin{aligned} x_1 &= r\sin(\varphi)\cos(\theta), & r &= \sqrt{x_1^2+x_2^2+x_3^2}, \\ x_2 &= r\sin(\varphi)\sin(\theta), & \varphi &= \cos^{-1}\left(\frac{x_3}{\sqrt{x_1^2+x_2^2+x_3^2}}\right), \\ x_3 &= r\cos(\varphi), & \theta &= \tan^{-1}(x_2/x_1). \end{aligned} \tag{3.51}$$

3.3.1 Cylindrical/polar stresses

Just as with Cartesian stresses we define polar stresses with reference to section cuts and force directions. For example, $\sigma_{r\theta}$ represents forces per unit area in the θ-direction on a section cut with normal vector $\boldsymbol{e}_r$. Such a surface is a cylinder. Similar meaning can be ascribed to other stress components taking $\{r, z, \theta\}$ for subscripts. Figure 3.14 shows some of these stresses on the standard polar/cylindrical element cut from a solid.

The equilibrium equations in cylindrical coordinates can be derived using differential element arguments (where the shape of the differential element is the same as the integration volume in cylindrical coordinates). The result of such an exercise is that

0 ⇒ δ

(3x)

[see eqns 3.38 -3.43]

$$\begin{aligned} \sum F_r &= \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} + b_r = 0 \\ \sum F_\theta &= \frac{\partial \sigma_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{\theta r}}{r} + b_\theta = 0 \\ \sum F_z &= \frac{\partial \sigma_{zr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{z\theta}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{zr}}{r} + b_z = 0 \end{aligned} \tag{3.52}$$

Note that moment equilibrium requires that $\sigma_{zr} = \sigma_{rz}$, $\sigma_{r\theta} = \sigma_{\theta r}$, and $\sigma_{z\theta} = \sigma_{\theta z}$. Thus just as with Cartesian stresses, the cylindrical/polar stresses are symmetric when placed in a matrix. In other words, the order of the subscripts does not matter.

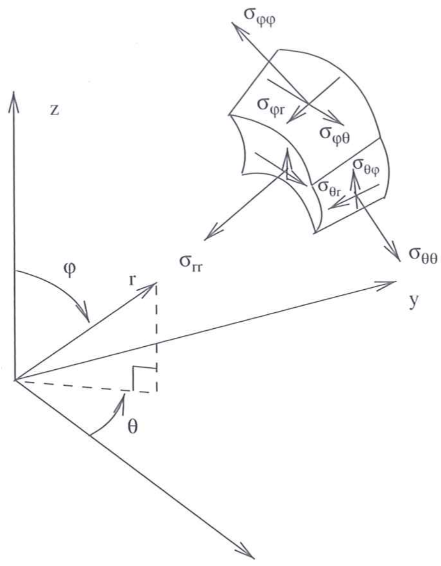


Fig. 3.15 Spherical stresses. Three-dimensional representation with some stresses labeled on the standard spherical element.

3.3.2 Spherical stresses

A similar interpretation of stress in spherical coordinates also holds. The first subscript always refers to the normal vector of the section cut, and the second the direction of the force. Some of the stresses are labeled in Fig. 3.15. Applying equilibrium to a differential element yields the relations:

$$
\begin{aligned}
\sum F_r =& \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\varphi}}{\partial \varphi} + \frac{1}{r\sin(\varphi)}\frac{\partial \sigma_{r\theta}}{\partial \theta} \\
&+ \frac{2\sigma_{rr} - \sigma_{\varphi\varphi} - \sigma_{\theta\theta} + \sigma_{r\varphi}\cot(\varphi)}{r} + b_r = 0 \\
\sum F_\varphi =& \frac{\partial \sigma_{\varphi r}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\varphi\varphi}}{\partial \varphi} + \frac{1}{r\sin(\varphi)}\frac{\partial \sigma_{\varphi\theta}}{\partial \theta} \\
&+ \frac{3\sigma_{\varphi r} + (\sigma_{\varphi\varphi} - \sigma_{\theta\theta})\cot(\varphi)}{r} + b_\varphi = 0 \\
\sum F_\theta =& \frac{\partial \sigma_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\varphi}}{\partial \varphi} + \frac{1}{r\sin(\varphi)}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} \\
&+ \frac{3\sigma_{r\theta} + 2\sigma_{\varphi\theta}\cot(\varphi)}{r} + b_\theta = 0
\end{aligned}
\qquad (3.53)
$$

0 ⇒ (3x)

[see eqns 3.38-3.43]

Note that moment equilibrium requires $\sigma_{\varphi r} = \sigma_{r\varphi}$, $\sigma_{r\theta} = \sigma_{\theta r}$, and $\sigma_{\varphi\theta} = \sigma_{\theta\varphi}$. Thus just as with Cartesian and cylindrical/polar stresses, the spherical stresses are symmetric when placed in a matrix. In other words, the order of the subscripts does not matter here either.

The matrix convention for reporting the components of the strain tensor in three dimensions is:

$$\varepsilon = \begin{bmatrix} \varepsilon_{xx} & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{zx} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_{yy} & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{yz} & \varepsilon_{zz} \end{bmatrix}. \tag{4.23}$$

4.3 Polar/cylindrical and spherical strain

Just as with stresses we can also define strains with respect to polar and spherical coordinate systems. Quantities such as $\varepsilon_{\theta\theta}$ represent normal strains in the θ-direction at a point; a quantity such as $\gamma_{\theta z}$ would represent a change in angle in the θ-z coordinate plane passing through a point.

Using constructions like those in the Cartesian case one finds that

$\frac{\partial u_r}{\partial r}$ 2x

$$\begin{bmatrix} \varepsilon_{rr} & \frac{1}{2}\gamma_{r\theta} & \frac{1}{2}\gamma_{rz} \\ & \varepsilon_{\theta\theta} & \frac{1}{2}\gamma_{\theta z} \\ \text{sym.} & & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} u_{r,r} & \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + r\frac{\partial (u_\theta/r)}{\partial r}\right) & \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \\ & \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right) & \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right) \\ \text{sym.} & & \frac{\partial u_z}{\partial z} \end{bmatrix}. \tag{4.24}$$

$$\begin{bmatrix} \varepsilon_{rr} & \frac{1}{2}\gamma_{r\varphi} & \frac{1}{2}\gamma_{r\theta} \\ & \varepsilon_{\varphi\varphi} & \frac{1}{2}\gamma_{\varphi\theta} \\ \text{sym.} & & \varepsilon_{\theta\theta} \end{bmatrix} = \begin{bmatrix} u_{r,r} & \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \varphi} + r\frac{\partial (u_\varphi/r)}{\partial r}\right) & \frac{1}{2}\left(\frac{1}{r\sin(\varphi)}\frac{\partial u_r}{\partial \theta} + r\frac{\partial (u_\theta/r)}{\partial r}\right) \\ & \frac{1}{r}\left(\frac{\partial u_\varphi}{\partial \varphi} + u_r\right) & \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \varphi} + \frac{1}{r\sin(\varphi)}\frac{\partial u_\varphi}{\partial \theta}\right) \\ \text{sym.} & & \frac{1}{r\sin(\varphi)}\frac{\partial u_\theta}{\partial \theta} + u_r/r + u_\varphi\cot(\varphi)/r \end{bmatrix}. \tag{4.25}$$

4.4 Number of unknowns and equations

If we look over our theory as developed we find in three dimensions that there are six strains, three displacements and nine stresses – giving a total of 18 possible unknown quantities. In two dimensions we find that we have three strains, two displacements, and four stresses – giving nine possible unknown quantities. The total number of equations at hand in z (4x) φ (4x) r (4x) θ (4x)

(6.8) Determine $\varepsilon_{\theta\theta}$, $\varepsilon_{\varphi\varphi}$, and ε_{rr} for a thin-walled spherical pressure vessel with internal pressure p. Assume homogeneous linear elastic material.

(6.9) How much pressure is required to thin the walls of a thin-walled spherical pressure vessel by 1%? Assume homogeneous elastic properties.

(6.10) You are to design a cylindrical pressure vessel with spherical end caps as shown below. The caps and the main part of the vessel are to have the same thickness (which you need to determine). The radius of the vessel and the length are specified to be R and L, respectively. The internal pressure is specified to be p. Design the vessel against yield using the following multi-axial criteria in the cylindrical portion of the vessel

$$\sqrt{(\sigma_r - \sigma_\theta)^2 + (\sigma_\theta - \sigma_z)^2 + (\sigma_z - \sigma_r)^2} \leq \sqrt{2}\sigma_Y$$

and

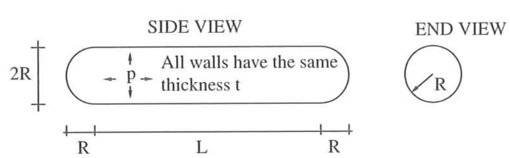
$$\sqrt{(\sigma_r - \sigma_\theta)^2 + (\sigma_\theta - \sigma_\varphi)^2 + (\sigma_\varphi - \sigma_r)^2} \leq \sqrt{2}\sigma_Y$$

in the spherical portion of the vessel. Allow for a safety factor of SF against yield. In choosing the wall thickness, choose a material that minimizes the weight of the vessel. Choose your material from Table 6.1.

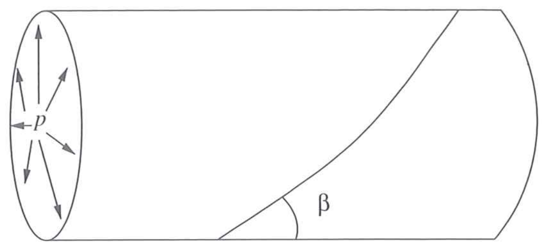
Table 6.1 Material selection list.

Material	E (GPa)	σ_Y (MPa)	Density (kg/m^3)
Al	70	470	2700
Mild Steel	200	260	7850
HS Steel	208	1300	7850
HTS graphite-5208 epoxy	183	1250	1550

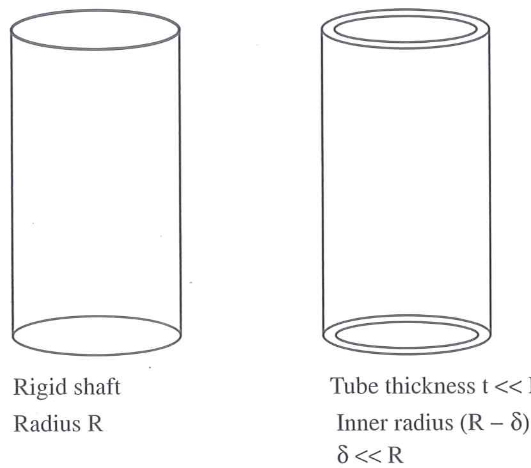
What is the best material and what is the required wall thickness?



(6.11) A pressure vessel of radius R, length L, and wall thickness t is constructed with a weld line at angle β. The dimensions are such that $t \ll R$ and $L \gg R$. The vessel is pressurized to a pressure p. What is the shear stress on the weld? Assume that the pressure vessel is closed on the ends.



(6.12) A linear elastic tube with radius $(R - \delta)$ is to be shrink-fit on a rigid shaft of radius R. The tube is first heated so as to expand its inner radius to at least R. It is then slipped over the rigid shaft, and is allowed to cool. Assuming that the shaft is well lubricated, find a relation for the contact pressure between the tube and the shaft. Your final answer should be given in terms the isotropic elastic material constants, misfit δ, thickness t, and rigid shaft radius R.



(6.13) Consider the following cross section of a composite cylindrical pressure vessel. Assume that the radius is much greater than the wall thicknesses. Assume that the strains are constants as a function of r. Thus, in particular, $\varepsilon_{zz}^{(1)} = \varepsilon_{zz}^{(2)}$ and the solution of the problem of non-circular torsion requires the explicit solution of the governing partial differential equations. Notwithstanding, there is one case where we can effectively deal with non-circular torsion without solving partial differential equations. This case arises when we make the added assumption of thinness; i.e. here we assume that the member in question is hollow with arbitrary cross-sectional shape but where the tube walls are assumed thin.

7.7.1 Equilibrium

Equilibrium in a thin-walled tube under torsion is governed by the same expression as we used for the circular bar, viz. eqn (7.12). This follows because the expression was derived solely using equilibrium concepts independent of the kinematic assumption. To connect the stresses on the section to the internal torque we can also use the same relations as before, viz. eqn (7.13). However, because of the thinness assumption we can actually reduce this expression to an explicit algebraic relation that gives the stress for a given torque, and vice versa. In contrast to the circular bar case, we can do this without employing a kinematic assumption. This occurs because we will be using a thinness assumption.

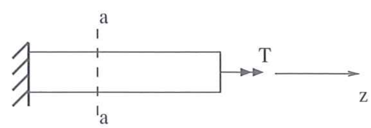


Fig. 7.35 Thin-walled tube under load.

7.7.2 Shear flow

Consider the tube shown in Fig. 7.35. It is subjected to a torque, T, at one end and is fixed at the other. Points on the tube cross-section will be described either by the angle α which is measured counter-clockwise from the x-axis, or by arclength s, also measured counter-clockwise from the x-axis; see Fig. 7.36. The origin of the x-y coordinate system is taken at the center of twist; the point about which the section rigidly rotates. Knowledge of this location is not needed in general, and its determination is beyond the scope of this text as it involves the solution of partial differential equations. The geometry of the tube is fully specified when one is given the two functions $r(\cdot)$ and $t(\cdot)$, where $r(\alpha)$ denotes the distance from the center of twist to the tube wall at angle α, and $t(\alpha)$ denotes the wall thickness at angle α. The wall thickness is measured perpendicular to the tube wall and not along the radial ray. The assumption of thinness amounts to assuming that

1/10

$$\frac{t(\alpha)}{r(\alpha)} < 10 \qquad (7.113)$$

for all α.

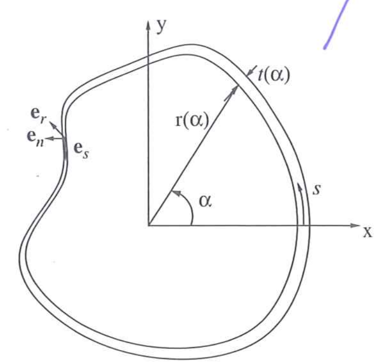


Fig. 7.36 Section *a-a* from Fig. 7.35; coordinate definitions for thin-walled tube geometry.

To support the applied torque there must be a shear stress on the cross-section. Further, because we are assuming the walls to be thin we can assume that the shear stress at each location α is constant across the thickness. We also have that the shear stress at each location α must be tangential to the wall. If the stresses were not tangential to the wall, then moment equilibrium expressed in terms of stresses would require shear stresses on the surface of the tube in contradiction to the manner

Solution

Figure 8.11 shows a free-body diagram of the beam with the supports removed. Force equilibrium in the x-direction tells us that $R_1 = 0$ and force equilibrium in the y-direction tells us that $R_2 = R_3$. Taking moments about any point along the axis of the beam shows that $R_2 = \bar{M}/L$ and, hence, $R_3 = \bar{M}/L$.

Internal force and moment diagrams can now be constructed by making successive section cuts and applying force/moment balance for different values of x, as shown in Fig. 8.11. For axial force balance, only a single section cut is necessary, since there are no distributed axial forces. This gives

$$0 = \sum F_x = R(x) + R_1 \qquad \Rightarrow \qquad R(x) = 0. \tag{8.16}$$

For vertical force balance we can also make just a single arbitrary cut. This yields

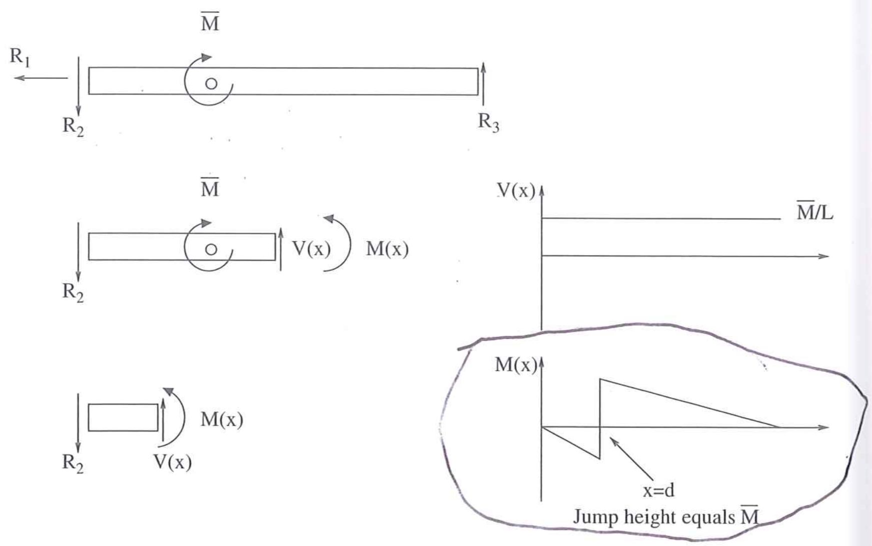
$$0 = \sum F_y = -\frac{\bar{M}}{L} + V(x) \qquad \Rightarrow \qquad V(x) = \frac{\bar{M}}{L}. \tag{8.17}$$

For moment equilibrium we need to make two different cuts – i.e. one before the applied point moment and one after. This gives

$$0 = \sum M_z = M(x) + \frac{\bar{M}}{L}x \qquad \Rightarrow M(x) = -\frac{\bar{M}}{L}x \qquad x < d \tag{8.18}$$

$$0 = \sum M_z = M(x) + \frac{\bar{M}}{L}x - \bar{M} \qquad \Rightarrow M(x) = \bar{M} - \frac{\bar{M}}{L}x \qquad x > d. \tag{8.19}$$

The resulting graphs are shown in Fig. 8.11.



Slopes should match

Fig. 8.11 Free-body diagram for Example 8.1 and internal force diagrams.

derive

Circular and Thin-Wall Torsion

7

The application of torsional loads to slender bars can be analyzed very effectively utilizing the techniques developed in Chapter 6. In particular, by making an appropriate kinematic assumptions we will be able to nicely analyze the behavior of twisted circular bars. For the case of bars with more general cross-sections the situation is a little more complex. But if we include the added requirement that the cross-section have the geometry of a thin-walled tube, then we will be able to take advantage of thinness to drive an effective analytical method for arbitrary thin-walled cross-sections. The subject of torsion of solid non-circular bars is left for more advanced courses where the solution of partial differential equations can be more comfortably tackled. Figures 7.1 and 7.2 shows some example situations where one finds structural members under torsional loads.

7.1 Circular bars: Kinematic assumption

The fundamental observation associated with the motion of circular bars in torsion is that the bar cross-sections remain planar and rigidly rotate under load. This observation is predominantly true over a very wide range of loadings, elastic and plastic behavior, and material inhomogeneities (with circular symmetry). Figure 7.3a shows a solid circular bar that is clamped at the base. At the top is a lever arm that allows one to apply a torque to the bar. Figure 7.3b shows a close-up of the square grid that has been painted on the surface of the bar. After the application of a torque to the bar the grid distorts as shown in Fig. 7.3c. Note that the horizontal lines remain planar and that there is a progressive rotation that is increasing at a constant rate as one moves up the bar. In the special case of torsion of an infinitely long bar, one can show via symmetry arguments that these observations are exact.

These physical observations can be turned into a usable kinematic relation by considering the geometric construction in Fig. 7.4, showing a circular bar under the action of two applied torques. First we slice the torsion bar at two elevations z and $z + \Delta z$. Due to the applied torque the bottom of the piece rotates an amount $\phi(z)$ and the top an amount $\phi(z + \Delta z)$. In the theory of torsion, the rotation of the section ϕ is similar to the displacement u in the theory of axial extension.

Fig. 7.1 An electric motor's shaft is loaded in torsion when in operation.

Fig. 7.2 Both the screwdriver shaft and the screw are loaded in torsion when driving the screw into the wood member.

where $Q(y_1) = \int_{A(y_1)} y\, dA$ is the first moment of the area above $y = y_1$. Thus,

$$\sigma_{xy}(x, y) = \frac{V(x)Q(y)}{Ib}. \tag{8.113}$$

Remarks:

(1) The product $\sigma_{xy}b$ is known as the shear flow, and is typically denoted by the letter q. Shear flow represents the force per unit length along the horizontal cut.

Example 8.15

Glued T-beam. Consider the T-beam shown in Fig. 8.27. The beam is constructed by gluing two boards together. If the maximum allowable shear stress in the glue is $\tau_{\max}$, how much force can the beam support. Assume that I and y_{glue} are given, and that the coordinate axes have been aligned with the neutral axis.

Solution

The internal shear force in the beam $V(x) = P$. The shear stress in the glue is thus

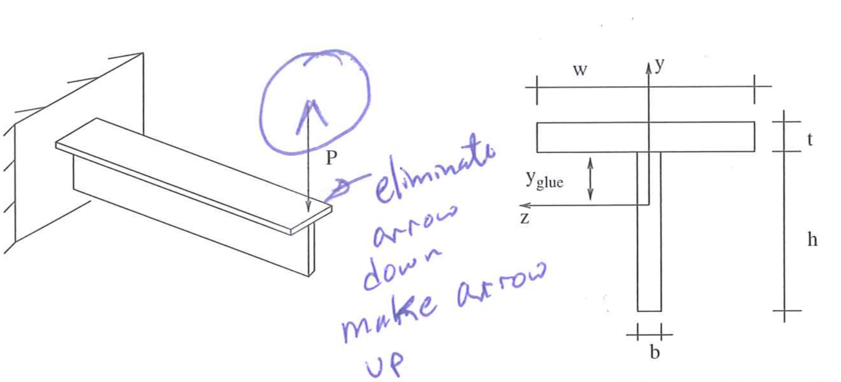
$$\sigma_{yx}(y_{\text{glue}}) = \frac{PQ(y_{\text{glue}})}{bI}. \tag{8.114}$$

The first moment of $A(y_{\text{glue}})$ is

$$Q(y_{\text{glue}}) = tw(y_{\text{glue}} + t/2) \tag{8.115}$$

This gives the requirement that

$$P < \frac{\tau_{\max} bI}{tw(y_{\text{glue}} + t/2)}. \tag{8.116}$$



Change direction of arrow

Fig. 8.27 Cantilevered T-beam with glue joint.

9.2.1 Traction vector method

Consider a point, P, in a body. At this point we will assume that we know the stress state relative to the x-y coordinate system; i.e., we will assume that we know the stress tensor components σ_{xx}, σ_{yy}, and σ_{xy}. We wish to determine the components of the stress tensor relative to the x'-y' coordinate system. Assuming the angle between the coordinate systems is θ, this means that we wish to find the normal and tangential components of the traction vector on planes with normal vectors $\boldsymbol{e}_{x'}$ and $\boldsymbol{e}_{y'}$. Referring to Fig. 9.2, these vectors are given by

$$\boldsymbol{e}_{x'} = \cos(\theta)\boldsymbol{e}_x + \sin(\theta)\boldsymbol{e}_y, \tag{9.12}$$

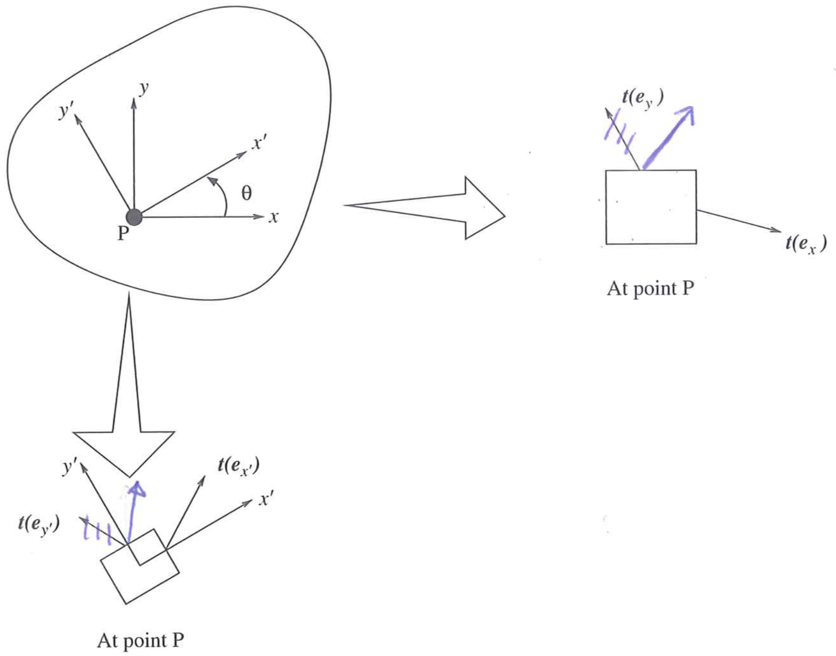
$$\boldsymbol{e}_{y'} = -\sin(\theta)\boldsymbol{e}_x + \cos(\theta)\boldsymbol{e}_y. \tag{9.13}$$

Using Cauchy's Law we know that the traction vector on any plane with normal vector $\boldsymbol{n}$ is given by

$$\boldsymbol{t}(\boldsymbol{n}) = \boldsymbol{\sigma}^T \boldsymbol{n}. \tag{9.14}$$

Thus we know that on the plane with normal vector $\boldsymbol{e}_{x'}$, the traction vector is

$$\begin{aligned} \begin{pmatrix} t_x(\theta) \\ t_y(\theta) \end{pmatrix} &= \begin{bmatrix} \sigma_{xx} & \sigma_{yx} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\theta)\sigma_{xx} + \sin(\theta)\sigma_{yx} \\ \cos(\theta)\sigma_{xy} + \sin(\theta)\sigma_{yy} \end{pmatrix}. \end{aligned} \tag{9.15}$$



Fix 2 arrow in figure

Fig. 9.2 Geometry used for deriving stress transformation rules.

coordinate system used to express the components of the tensor. In terms of the components this implies:

$$\mathrm{I}_\sigma = \mathrm{trace}[\boldsymbol{\sigma}] \quad = \sigma_{xx} + \sigma_{yy} \quad = \sigma_{x'x'} + \sigma_{y'y'} \tag{9.26}$$

$$\mathrm{II}_\sigma = \det[\boldsymbol{\sigma}] \quad = \sigma_{xx}\sigma_{yy} \cancel{+} \sigma_{xy}^2 \quad = \sigma_{x'x'}\sigma_{y'y'} \cancel{+} \sigma_{x'y'}^2. \tag{9.27}$$

minus sign

$-/ 2x$

The invariants are useful for double-checking numerical computations. They are also useful as they represent intrinsic properties of the state of stress independent of coordinate system.

(4) For three-dimensional stress transformations we have a result similar to the one presented except, that the rotation matrix is replaced by a three-dimensional rotation matrix. In three dimensions the invariants are defined as

$$\mathrm{I}_\sigma = \mathrm{trace}[\boldsymbol{\sigma}] \tag{9.28}$$

$$\mathrm{II}_\sigma = \cancel{\frac{1}{2}[\mathrm{trace}(\boldsymbol{\sigma}^2) - (\mathrm{trace}(\boldsymbol{\sigma}))^2]} \tag{9.29}$$

$$\mathrm{III}_\sigma = \det[\boldsymbol{\sigma}]. \tag{9.30}$$

math bold

$\frac{1}{2}[(\mathrm{trace}[\sigma])^2 - \mathrm{trace}(\sigma^2)]$

Example 9.1

Transformation of stress. Consider a welded plate, as shown in Fig. 9.3, with applied loads such that it is in an homogeneous state of two-dimensional stress relative to the x-y axes of

$$\begin{bmatrix} 100 & 50 \\ 50 & 20 \end{bmatrix}_{xy} \text{ksi.} \tag{9.31}$$

Find the state of stress in the plate relative to a set of axes aligned with the weld.

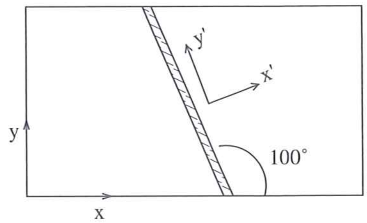


Fig. 9.3 Uniformly stressed plate with a weld.

Solution
Select the x' axis to be orthogonal to the weld line. Then we have that $\theta = 10 \times \pi/180$ rad. Inserting into the double angle transformation equations, we find

$$\sigma_{x'x'} = 60 + 40\cos(2\theta) + 50\sin(2\theta) \tag{9.32}$$

$$\sigma_{y'y'} = 60 - 40\cos(2\theta) - 50\sin(2\theta) \tag{9.33}$$

$$\sigma_{x'y'} = \quad - 40\sin(2\theta) + 50\cos(2\theta), \tag{9.34}$$

which results in

$$\begin{bmatrix} 114.7 & 33.3 \\ 33.3 & 5.3 \end{bmatrix}_{x'y'} \text{ksi.} \tag{9.35}$$

Remarks:

(1) The matrix answer has been subscripted with the coordinate frame. This is to remind us that the components in the matrix are relative to the x'-y' frame and not the x-y coordinate frame.

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} \begin{pmatrix} \cos(\theta_p) \\ \sin(\theta_p) \end{pmatrix} = \sigma_1 \begin{pmatrix} \cos(\theta_p) \\ \sin(\theta_p) \end{pmatrix} \tag{9.48}$$

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} \begin{pmatrix} -\sin(\theta_p) \\ \cos(\theta_p) \end{pmatrix} = \sigma_2 \begin{pmatrix} -\sin(\theta_p) \\ \cos(\theta_p) \end{pmatrix}. \tag{9.49}$$

This reveals that the principal values are nothing more that the eigenvalues of the stress tensor and that the eigenvectors of the stress tensor correspond to the principal directions. The classical eigenvalue problem is usually written as

$$(\boldsymbol{\sigma} - \lambda \mathbf{1})\boldsymbol{n} = \mathbf{0}, \tag{9.50}$$

where λ is the eigenvalue and $\boldsymbol{n}$ is the eigenvector. The condition for a non-trivial solution to these homogeneous equations is that $\det(\boldsymbol{\sigma} - \lambda\mathbf{1}) = 0$. This (in two dimensions) produces a quadratic polynomial in λ (the characteristic polynomial):

$$-\lambda^2 + \mathrm{I}_\sigma \lambda - \mathrm{II}_\sigma = 0. \tag{9.51}$$

The two roots of this equation are the principal values.

Remarks:

(1) This observation about the connection between principal values and eigenvalues also holds true in three dimensions. In three dimensions the characteristic polynomial is given as

$$-\lambda^3 + \mathrm{I}_\sigma \lambda^2 - \mathrm{II}_\sigma \lambda + \mathrm{III}_\sigma = 0 \tag{9.52}$$

and there will be three principal values ($\sigma_1 \geq \sigma_2 \geq \sigma_3$) and three principal directions.

(2) While it may seem more complex to discuss the eigenvalues of a tensor, this is in practice the easiest way to compute the principal values of a general three-dimensional state of stress. This especially holds true due to efficient algorithms for numerically computing eigenvalues and eigenvectors.

(3) By the properties of symmetric tensors, we also have the result that the principal directions will always be orthogonal to each other – the eigenvectors of symmetric matrices can always be chosen to be orthogonal.

(4) The maximum shear stress is given by $\tau_{\max} = \frac{1}{2}(\sigma_1 - \sigma_2)$ in two dimensions and $\tau_{\max} = \frac{1}{2}(\sigma_1 - \sigma_3)$ in three dimensions.

Example 9.2

Principal stresses. Given a state of stress

1.00
3.00

$$\begin{bmatrix} \cancel{1.0} & 10.0 \\ 10.0 & \cancel{3.0} \end{bmatrix} \text{MPa} \tag{9.53}$$

find the principal values and principal directions.

Solutions

Compute the eigenvalues as the roots of the characteristic polynomial

1.00 (3x)

3.00 (3x)

$$\det\begin{bmatrix} \cancel{1.0}-\lambda & 10.0 \\ 10.0 & \cancel{3.0}-\lambda \end{bmatrix} = 0 \tag{9.54}$$

$$(1.0-\lambda)(3.0-\lambda) - 100.0 = 0. \tag{9.55}$$

Using the quadratic formula we find

$$\sigma_1 = 12.0 \text{ MPa} \tag{9.56}$$

$$\sigma_2 = \cancel{-8.0} \text{ MPa}. \tag{9.57}$$

-8.05

The principal directions are given by the eigenvectors which are found by solving the linear equations

$$\begin{bmatrix} \cancel{1.0}-12.0 & 10.0 \\ 10.0 & \cancel{3.0}-12.0 \end{bmatrix} \begin{pmatrix} n_x \\ n_y \end{pmatrix} = 0 \tag{9.58}$$

and

$$\begin{bmatrix} \cancel{1.0}+\cancel{8.0} & 10.0 \\ 10.0 & \cancel{3.0}+\cancel{8.0} \end{bmatrix} \begin{pmatrix} n_x \\ n_y \end{pmatrix} = 0. \tag{9.59}$$

8.05 (2x)

The first principal direction is found to be

$$\begin{pmatrix} \cancel{0.6710} \\ \cancel{0.7415} \end{pmatrix} \tag{9.60}$$

0.671

0.742

and the second

$$\begin{pmatrix} \cancel{-0.7415} \\ \cancel{0.6710} \end{pmatrix}. \tag{9.61}$$

-0.742

0.671

To compute the principal angle we can always use the relation $\theta_p = \cos^{-1}(\boldsymbol{e}_x \cdot \boldsymbol{n}_1) = 0.8352$ rad, where $\boldsymbol{n}_1$ is the first principal direction.

9.2.4 Mohr's circle of stress

Mohr's circle is a graphical device that allows one to have a visual picture of all the normal and shear stress combinations which are possible by a change of coordinate basis. The device is usually applied to two-dimensional states of stress. It can, however, also be applied to three-dimensional states of stress when one of the principal stresses is known *a priori*.

Mohr's circle (of stress) is based upon the following writing of the transformation equations:

$$\begin{aligned}
\varepsilon_{x'x'} &= \left(\cos(\theta)\frac{\partial}{\partial x} + \sin(\theta)\frac{\partial}{\partial y}\right)(u_x\cos(\theta) + u_y\sin(\theta)) \\
&= \cos^2(\theta)\frac{\partial u_x}{\partial x} + \cos(\theta)\sin(\theta)\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) + \sin^2(\theta)\frac{\partial u_y}{\partial y} \\
&= \cos^2(\theta)\varepsilon_{xx} + 2\cos(\theta)\sin(\theta)\varepsilon_{xy} + \sin^2(\theta)\varepsilon_{yy}. \qquad (9.70)
\end{aligned}$$

Similar expressions can be derived for $\varepsilon_{y'y'} = \partial u_{y'}/\partial y'$ and $\varepsilon_{x'y'} = \frac{1}{2}(\partial u_{x'}/\partial y' + \partial u_{y'}/\partial x')$. When combined, we find

$$\begin{bmatrix} \varepsilon_{x'x'} & \varepsilon_{x'y'} \\ \varepsilon_{y'x'} & \varepsilon_{y'y'} \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}. \qquad (9.71)$$

Remarks:

(1) Equation (9.71) is identical in form to the transformation rule for stresses given in eqn (9.19).

(2) Just as for stresses, the transformation rules for strain are often expressed using the double angle formulae:

$$\varepsilon_{x'x'} = \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} + \frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\cos(2\theta) + \varepsilon_{xy}\sin(2\theta) \qquad (9.72)$$

$$\varepsilon_{y'y'} = \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} - \frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\cos(2\theta) - \varepsilon_{xy}\sin(2\theta) \qquad (9.73)$$

$$\varepsilon_{x'y'} = -\frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\sin(2\theta) + \varepsilon_{xy}\cos(2\theta). \qquad (9.74)$$

(3) For the same reason as with the stress tensor, the strain tensor also possesses two invariants in two dimensions. These are $\mathrm{I}_\varepsilon = \mathrm{trace}[\boldsymbol{\varepsilon}]$ and $\mathrm{II}_\varepsilon = \det[\boldsymbol{\varepsilon}]$ with associated characteristic polynomial:

$$-\lambda^2 + \mathrm{I}_\varepsilon\lambda - \mathrm{II}_\varepsilon = 0. \qquad (9.75)$$

In three dimensions $\mathrm{I}_\varepsilon = \mathrm{trace}[\boldsymbol{\varepsilon}]$, $\mathrm{II}_\varepsilon = \cancel{\frac{1}{2}(\mathrm{trace}[\boldsymbol{\varepsilon}^2] - (\mathrm{trace}[\boldsymbol{\varepsilon}])^2)}$, and $\mathrm{III}_\varepsilon = \det[\boldsymbol{\varepsilon}]$ with associated characteristic polynomial:

mathbold

$\frac{1}{2}((\mathrm{trace}[\varepsilon])^2 - \mathrm{trace}[\varepsilon^2])$

$$-\lambda^3 + \mathrm{I}_\varepsilon\lambda^2 - \mathrm{II}_\varepsilon\lambda + \mathrm{III}_\varepsilon = 0. \qquad (9.76)$$

9.3.1 Maximum normal and shear strains

As with stresses, we can speak of principal strains. In two dimensions, principal strains will represent the maximum and minimum normal strains in the plane. The mathematics of computing these values follows exactly our developments for stress. In this regard we can follow any of the approaches shown for stresses.

$$= \frac{1}{2}\left(\sigma_{x'x'}\varepsilon_{x'x'} + \sigma_{y'y'}\varepsilon_{y'y'}\right)$$
$$= \frac{1}{2}\left(\sigma_{x'x'}\left[\frac{\sigma_{x'x'}}{E} - \frac{\nu}{E}\sigma_{y'y'}\right] + \sigma_{y'y'}\left[\frac{\sigma_{y'y'}}{E} - \frac{\nu}{E}\sigma_{x'x'}\right]\right)$$
$$= \frac{1}{2}c^2\frac{2+2\nu}{E}. \tag{9.90}$$

If we equate the two results, which describe the exact same physical state, then we see that

$$G = \frac{E}{2(1+\nu)}. \tag{9.91}$$

9.4 Multi-axial failure criteria

In the chapters up to this point we have analyzed systems dominated by a single component of the stress tensor. For this reason, we have been able to simply determine the load-carrying capacity of different systems by requiring that

$$|\sigma| \leq \sigma_Y \tag{9.92}$$

for systems that are dominated by a single normal stress, or that

$$|\tau| \leq \tau_Y \tag{9.93}$$

for systems that are dominated by a single shear stress τ. When we have a multi-axial state of stress (more that one non-zero stress component), then we are faced with a new problem. Do we simply enforce eqns (9.92) and (9.93) component-by-component or should we do something else? As it turns out, component-by-component enforcement of eqns (9.92) and (9.93) does not comport with experimental experience. Further, it has the unfortunate side-effect of being coordinate system dependent. In other words, component by component enforcement of eqns (9.92) and (9.93) will not be able to predict material yielding/failure independent of the coordinate system used to express the stress components. What we would like is a criterion that evaluates some function of the stress tensor and outputs a value indicating yield/failure or not yield/failure. The function should be independent of coordinate system for it to be physically meaningful. Thus we would like something of the form

$$f(\sigma) \leq \text{critial value.} \tag{9.94}$$

For polycrystalline metallic materials at room temperature the two most common criteria of this form are Tresca's yield condition and the Henky–von Mises yield condition; these are both discussed next. Criteria for the multi-axial failure/yield of brittle materials and materials with different properties in tension and compression is left for more advanced texts. ck

9.4.2 Henky–von Mises condition

c/ 4x

The Henky–von Mises condition for yield is based upon an energetic idea. Their idea was that yield in a multi-axial state of stress will occur when the strain energy density reaches a critical value. The critical value is found by computing the strain energy density in a uniaxial test specimen at the moment of initial yield. Because yield is generally observed to be independent of pressure, the Henky–von Mises condition involves only the part of the strain energy associated with the deviatoric stresses. The strain energy density is given by:

$$w = \frac{1}{2}(\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{zz}\varepsilon_{zz} + \sigma_{xy}\gamma_{xy} + \sigma_{yz}\gamma_{yz} + \sigma_{zx}\gamma_{zx}). \tag{9.102}$$

If we substitute in for the strains using Hooke's Law and separate the pressure contributions from the deviatoric contributions, then we find that

$$\begin{aligned} w = & \underbrace{\frac{1+\nu}{2E}(s_{xx}^2 + s_{yy}^2 + s_{zz}^2 + 2s_{xy}^2 + 2s_{yz}^2 + 2s_{zx}^2)}_{w_{\text{dev}}} \\ & + \underbrace{\frac{3(1-2\nu)}{2E}p^2}_{w_{\text{vol}}}. \end{aligned} \tag{9.103}$$

The Henky–von Mises condition then provides that

$$w_{\text{dev}} \leq w_{\text{dev}}^{1-\text{D}}, \tag{9.104}$$

where $w_{\text{dev}}^{1-\text{D}}$ is the calibration constant determined from a uniaxial test. In a uniaxial test at yield the stress is given as:

$$\begin{bmatrix} \sigma_Y & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{9.105}$$

Thus the pressure is $p = \frac{1}{3}\sigma_Y$ and the deviatoric stress is given as

$$\begin{bmatrix} \frac{2}{3}\sigma_Y & 0 & 0 \\ 0 & -\frac{1}{3}\sigma_Y & 0 \\ 0 & 0 & -\frac{1}{3}\sigma_Y \end{bmatrix}. \tag{9.106}$$

This gives

$$w_{\text{dev}}^{1-\text{D}} = \frac{2(1+\nu)}{6E}\sigma_Y^2. \tag{9.107}$$

So we have as our yield criteria

$$(s_{xx}^2 + s_{yy}^2 + s_{zz}^2 + 2s_{xy}^2 + 2s_{yz}^2 + 2s_{zx}^2) \leq \frac{2}{3}\sigma_Y^2. \tag{9.108}$$

This expression can be made a little more convenient by expressing it directly in terms of the stress components instead of the components of the deviatoric stress; this can be done using the definition of the stress deviator, eqn (9.97). The net result is given as

$$\frac{1}{2}\left((\sigma_{xx}-\sigma_{yy})^2+(\sigma_{yy}-\sigma_{zz})^2+(\sigma_{zz}-\sigma_{xx})^2\right) \\ +3(\sigma_{xy}^2+\sigma_{yz}^2+\sigma_{zx}^2)\leq\sigma_Y^2. \qquad (9.109)$$

As long as the strict inequality is satisfied, the multi-axial stress state represents an elastic state of stress. When the equality is satisfied, yield starts.

Remarks:

(1) One advantage of the Henky–von Mises condition is that there is no need to determine Mohr's circles, principal stresses, or maximum shears. It works directly with the stress components. This point makes it especially easier to use in computer programs for automated stress analysis.

(2) Though not obvious, as written, the Henky–von Mises condition is independent of coordinate system.

(3) By construction, the Henky–von Mises condition is independent of pressure.

(4) Equation (9.109) is given in terms of an x-y-z Cartesian coordinate system. It is also valid for any other orthonormal coordinate system. For example, for cylindrical/polar coordinates or for spherical coordinates under the substitutions $(x,y,z)\to(r,\theta,z)$ and $(x,y,z)\to(r,\varphi,\theta)$, respectively.

(5) The Henky–von Mises condition for yield is a model just as the Tresca condition is a model for yield. They are both models for the same phenomena and will differ slightly in their predictions. One way to appreciate their differences is to consider a state of plane stress in principal coordinates:

$$\begin{bmatrix}\sigma_1 & 0 & 0\\ 0 & \sigma_2 & 0\\ 0 & 0 & 0\end{bmatrix}. \qquad (9.110)$$

Applying the Henky–von Mises condition to this state of stress we see that

$$(\sigma_1-\sigma_2)^2+\sigma_2^2+\sigma_1^2\leq 2\sigma_Y^2. \qquad (9.111)$$

This expression tells us that in the σ_1-σ_2 plane that the set of elastic stress states is contained in an elliptical region. The application of Tresca's condition to this state of stress gives a hexagonal region of elastic states in the same plane. The situation is sketched in Fig. 9.19. As can be seen, both criteria are in reasonable agreement with each other. In terms of ability to accurately model data, both criteria provide decent accuracy for polycrystalline metals at room temperature.

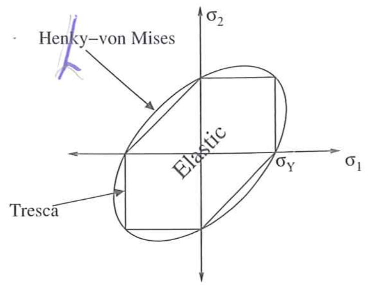


Fig. 9.19 Comparison of Henky–von Mises condition with Tresca's criteria in plane stress.

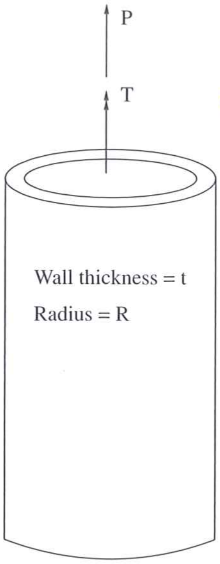


Fig. 9.20 Thin-walled tube in torsion and axial loading.

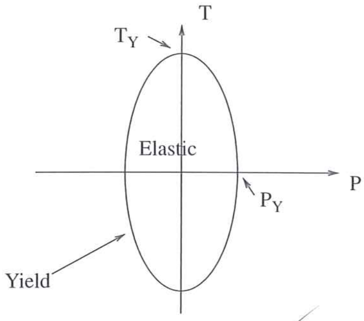


Fig. 9.21 Elastic zone for extension and twist of a thin-walled tube.

Example 9.11

Thin-walled tube in axial and torsional loading. Consider the thin-walled tube shown in Fig. 9.20. Determine the limits on P and T for combined states of loading according to the Henky–von Mises condition.

Solution

There are two non-zero stresses in the tube. Assuming that the axis of the tube is in the z direction, the axial force gives rise to

$$\sigma_{zz} = \frac{P}{A} = \frac{P}{2\pi Rt}. \tag{9.112}$$

The torque gives rise to a shear stress:

$$\sigma_{z\theta} = \frac{q}{t} = \frac{T}{2A_{\text{enclosed}}t} = \frac{T}{2\pi R^2 t}. \tag{9.113}$$

In matrix form we have:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & T/2\pi R^2 t \\ 0 & T/2\pi R^2 t & P/2\pi Rt \end{bmatrix}_{r\theta z}. \tag{9.114}$$

Applying eqn (9.109) in cylindrical coordinates gives:

$$\frac{1}{2}\left\{\left(\frac{P}{2\pi Rt}\right)^2 + (0)^2 + \left(-\frac{P}{2\pi Rt}\right)^2\right\} + 3\left(\frac{T}{2\pi R^2 t}\right)^2 \leq \sigma_Y^2 \tag{9.115}$$

$$\frac{P^2}{(2\pi Rt\sigma_Y)^2} + \frac{T^2}{(\frac{2}{\sqrt{3}}\pi R^2 t\sigma_Y)^2} \leq 1 \tag{9.116}$$

$$\frac{P^2}{P_Y^2} + \frac{T^2}{T_Y^2} \leq 1, \tag{9.117}$$

where $P_Y = 2\pi Rt\sigma_Y$ and $T_Y = \frac{2}{\sqrt{3}}\pi R^2 t\sigma_Y$. The loci of points in the P-T plane that correspond to yield are shown in Fig. 9.21. The interior of the ellipse corresponds to elastic states of loading, and yield occurs for any (P, T) combination on the ellipse.

Remarks:

(1) Notice that if one biases a torsional loading with any amount of axial force, then the permissible amount of torque is decreased. Likewise if one biases an axial loading with any amount of torque, then the permissible amount of axial force decreases.

Example 9.12

Thin-walled cylindrical pressure vessel. Consider a thin-walled cylindrical pressure vessel of radius R and wall thickness t. How much pressure ck 3x

can be applied before yield takes place according to the Henky–von Mises condition?

Solution
The state of stress in a cylindrical pressure vessel (see Section 6.2.1) is

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{pR}{t} & 0 \\ 0 & 0 & \frac{pR}{2t} \end{bmatrix}_{r\theta z} . \tag{9.118}$$

Thus by eqn (9.109) we have that

$$\left(\frac{pR}{t} - \frac{pR}{2t}\right)^2 + \left(\frac{pR}{2t} - 0\right)^2 + \left(\frac{pR}{t} - 0\right)^2 \leq 2\sigma_Y^2 \tag{9.119}$$

$$p \leq \frac{t}{R}\frac{2}{\sqrt{3}}\sigma_Y . \tag{9.120}$$

Example 9.13

Relation of τ_Y to σ_Y according to the Henky-von Mises condition. Consider an experiment that produces a state of yield in pure shear. Find the relation between a measured yield stress in shear, τ_Y, and the yield stress in tension, σ_Y.

Solution
At yield, the stress is given by

$$\begin{bmatrix} 0 & \tau_Y & 0 \\ \tau_Y & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \tag{9.121}$$

Evaluating the Henky–von Mises condition for this state of stress gives

$$3\tau_Y^2 = \sigma_Y^2 . \tag{9.122}$$

Thus,

$$\tau_Y = \frac{\sigma_Y}{\sqrt{3}} . \tag{9.123}$$

Remarks:

(1) Note that this relation is different than the one we derived in the section on Tresca's condition. The relation between the yield stress in shear and the yield stress in uniaxial tension is model-dependent – hence the difference.

Chapter summary

- Vector transformation rules:

$$\begin{pmatrix} F_{x'} \\ F_{y'} \end{pmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{pmatrix} F_x \\ F_y \end{pmatrix}$$

- Stress transformation rules:

$$\begin{bmatrix} \sigma_{x'x'} & \sigma_{x'y'} \\ \sigma_{y'x'} & \sigma_{y'y'} \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

  - Double-angle form:

$$\sigma_{x'x'} = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2} \cos(2\theta) + \sigma_{xy} \sin(2\theta)$$

$$\sigma_{y'y'} = \frac{\sigma_{xx} + \sigma_{yy}}{2} - \frac{\sigma_{xx} - \sigma_{yy}}{2} \cos(2\theta) - \sigma_{xy} \sin(2\theta)$$

$$\sigma_{x'y'} = -\frac{\sigma_{xx} - \sigma_{yy}}{2} \sin(2\theta) + \sigma_{xy} \cos(2\theta)$$

- Principal angle (maximum and minimum normal stresses with no shear)

$$\tan(2\theta_p) = \frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}}$$

- Maximum shear orientation (normal stress take their mean value)

$$\tan(2\theta_s) = -\frac{\sigma_{xx} - \sigma_{yy}}{2\sigma_{xy}}$$

  - Angle relation: $\theta_p - \theta_s = \pi/4$
  - Principal value form: $\tau_{\max} = (\sigma_1 - \sigma_3)/2$
- Mohr's circle: The center of the circle is the mean normal stress. The radius is the distance from the center to the point $(\sigma_{xx}, \sigma_{xy})$
- Three-dimensional case: Compute the eigenvalues and eigenvectors of $\boldsymbol{\sigma}$ from the characteristic polynomial:

$$-\lambda^3 + \mathrm{I}_\sigma \lambda^2 - \mathrm{II}_\sigma \lambda + \mathrm{III}_\sigma = 0$$

$\frac{1}{2}[(\mathrm{trace}(\sigma))^2 - \mathrm{trace}(\sigma^2)]$

  where $\mathrm{I}_\sigma = \mathrm{trace}[\sigma]$, $\mathrm{II}_\sigma = \cancel{\frac{1}{2}[\mathrm{trace}(\boldsymbol{\sigma}^2) - (\mathrm{trace}(\boldsymbol{\sigma}))^2]}$, $\mathrm{III}_\sigma = \det[\sigma]$

math bold

- Strains transform the same way as stresses, but one needs to use the tensorial shear strain instead of the engineering shear strain.
- Tresca's yield condition: $\tau_{\max} \leq \tau_Y$
- Deviatoric stress: $s = \boldsymbol{\sigma} - p\mathbf{1}$; pressure $p = \frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$

- Henky–von Mises yield condition:

$$\frac{1}{2}\left((\sigma_{xx}-\sigma_{yy})^2+(\sigma_{yy}-\sigma_{zz})^2+(\sigma_{zz}-\sigma_{xx})^2\right)$$
$$+3(\sigma_{xy}^2+\sigma_{yz}^2+\sigma_{zx}^2)\leq\sigma_Y^2$$

ch

Exercises

(9.1) The state of stress at a point in a solid is given as

$$\begin{bmatrix}10 & 5\\ 5 & 7\end{bmatrix}_{xy}\ \text{MPa}.$$

Consider a plane passing through this point with normal vector $\boldsymbol{n}=(1/\sqrt{2})\boldsymbol{e}_x+(1/\sqrt{2})\boldsymbol{e}_y$. What are the normal and shear stresses on this plane?

(9.2) Given the following two-dimensional state of stress, find the principal values and directions.

$$\begin{bmatrix}60 & 80\\ 80 & -90\end{bmatrix}\ \text{MPa}$$

(9.3) Given the following two-dimensional state of stress, find the maximum shear and sketch the state of stress on a properly oriented element.

$$\begin{bmatrix}60 & 80\\ 80 & -90\end{bmatrix}\ \text{MPa}$$

(9.4) Given the following two-dimensional state of stress, find the angles of rotation which cause the normal stress in the x' direction to be zero.

$$\begin{bmatrix}60 & 80\\ 80 & -90\end{bmatrix}\ \text{MPa}$$

(9.5) Using an eigenvalue technique, find the principal values and directions for the following state of stress.

$$\begin{bmatrix}10 & -50\\ -50 & 5\end{bmatrix}\ \text{ksi}$$

(9.6) Find the principal values of stress for the following three-dimensional state of stress. Sketch the three circles of stress on a Mohr diagram.

$$\begin{bmatrix}10 & -50 & 0\\ -50 & 5 & 0\\ 0 & 0 & 60\end{bmatrix}\ \text{ksi}$$

(9.7) Find the principal values and directions for the following three-dimensional state of stress. Sketch the three circles of stress on a Mohr diagram.

$$\begin{bmatrix}10 & -50 & 2\\ -50 & 5 & 0\\ 2 & 0 & 60\end{bmatrix}\ \text{ksi}$$

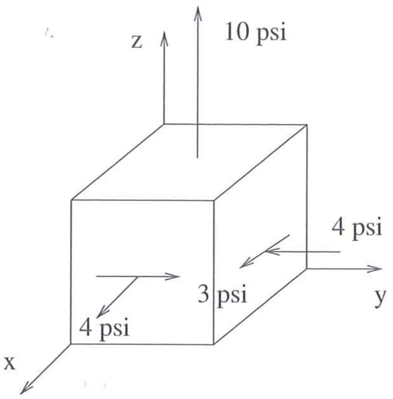
(9.8) Find the principal values of stress for the following three-dimensional state of stress. Sketch the three circles of stress on a Mohr diagram.

$$\begin{bmatrix}10 & 0 & 10\\ 0 & 5 & 0\\ 10 & 0 & 60\end{bmatrix}\ \text{ksi}$$

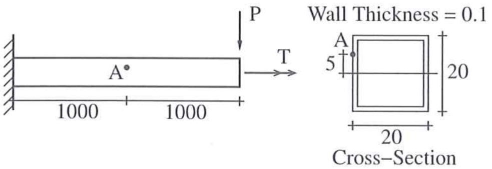
(9.9) Find the principal values of stress for the following three-dimensional state of stress. Sketch the three circles of stress on a Mohr diagram.

$$\begin{bmatrix}10 & 2 & 1\\ 2 & 5 & 0\\ 1 & 0 & 6\end{bmatrix}\ \text{MPa}$$

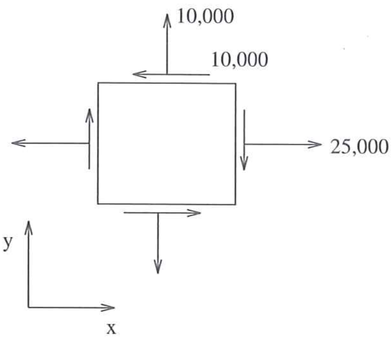
(9.10) For the state of stress that follows, find the principal stresses and principal directions. Draw the principal stresses on a properly oriented three-dimensional element.



(9.11) For the beam shown below, determine the principal stresses at point A and show them on a properly oriented element. What is the maximum shear stress at point A. Assume $P = 1$ and $T = 1000$ in consistent units.



(9.12) For the state of stress shown, determine and show the magnitude and direction of the principal stresses. How large is the maximum shear stress (not necessarily in the $x - y$ plane)? In what plane does it occur? Note, $\sigma_z = \tau_{xz} = \tau_{yz} = 0$. Assume stress units of kPa.

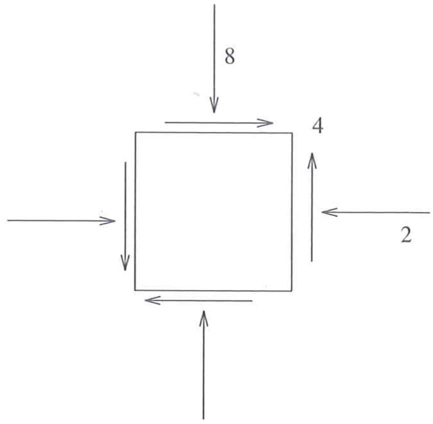


(9.13) Consider the two-dimensional state of stress shown; assume units of psi.

(a) Determine the principal stresses and show them on a properly oriented element.

(b) Determine the maximum shear stress in the plane and show this state on a properly oriented element.

(c) If the third principal stress is given as 10 psi, what is τ_{max}?



(9.14) For the two-dimensional state of plane stress

$$\sigma = \begin{bmatrix} -a & -a \\ -a & a \end{bmatrix},$$

find the principal stresses, principal angle, maximum shear, and maximum shear angle. Show your results on properly oriented elements.

(9.15) The displacement field in a structure has been measured to be

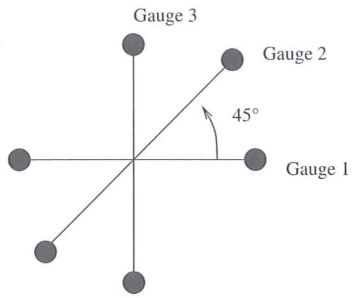
$$u_x = Ax + Ay$$
$$u_y = Ax$$

where A is a given constant. What is the maximum normal strain in the structure?

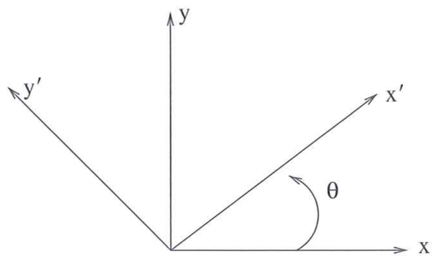
(9.16) As shown, a 0-45-90 strain-gauge rosette is applied to a thin plate of a material with Young's modulus $E = 12{,}500$ ksi and shear modulus $G = 5{,}000$ ksi, produces the following readings: $\varepsilon_{0^\circ} = 200 \times 10^{-6}$, $\varepsilon_{45^\circ} = 120 \times 10^{-6}$, $\varepsilon_{90^\circ} = -160 \times 10^{-6}$.

(a) Find the principal stresses and the maximum shear stress (assume plane stress).

(b) Check if this is a possible state of stress if the material is elastic–perfectly plastic, with a yield stress $\sigma_Y = 5$ ksi, according to the Henky–von Mises yield criterion.



(9.17) Consider two coordinate systems that are rotated an amount θ with respect to each other. Derive an expression for $\varepsilon_{x'y'}$ in terms of ε_{xx}, ε_{yy}, ε_{xy}, and θ.



(9.18) Drive the expressions for the strain components in the x-y coordinate system in terms of the output of a 0-60-120 strain-gauge rosette; i.e. knowing the normal strains in the three directions oriented 0, 60, and 120 degrees relative to the x-axis find expressions for ε_{xx}, ε_{yy}, and ε_{xy}.

(9.19) Derive the transformation equation for the normal strain in the y' direction:

$$\varepsilon_{y'y'} = \varepsilon_{yy}\cos^2(\theta) + \varepsilon_{xx}\sin^2(\theta) - 2\varepsilon_{xy}\sin(\theta)\cos(\theta)$$

(9.20) Show that the principal axes of stress and strain coincide for isotropic linear elastic materials; see Chapter 5.

(9.21) Consider a linear elastic isotropic material with $E = 100$ MPa and $\nu = -0.1$. The state of strain is known at a particular point to be

$$\begin{bmatrix} 10 & -50 \\ -50 & 5 \end{bmatrix}_{xy} \mu\text{strain}$$

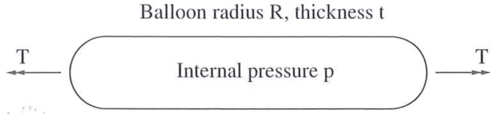
Find the principal stresses and the principal angle. Assume plane stress.

(9.22) Consider a linear elastic isotropic material with $E = 200$ MPa and $\nu = 0.3$. The state of stress is known at a particular point to be

$$\begin{bmatrix} 100 & -60 \\ -60 & 70 \end{bmatrix}_{xy} \text{MPa}$$

Find the principal strains and the principal angle. Assume plane stress.

(9.23) Shown below is a thin-walled cylindrical balloon with internal pressure p, radius R, and thickness t. Due to the thinness of the balloon walls they are incapable of supporting any compressive stresses. Find a relation for the maximum torque that the balloon will support. Express your answer in terms of p and R.



(9.24) Consider the following state of stress:

$$k\begin{bmatrix} 10 & -6 & 5 \\ -6 & 7 & 5 \\ 5 & 5 & 0 \end{bmatrix} \text{MPa.}$$

2x

ch

Determine the value of the parameter k for yield according to the Henky–von Mises condition. Assume $\sigma_Y = 100$ MPa.

(9.25) Is the following stress state elastic according to Tresca's condition:

$$\begin{bmatrix} 100 & -6 & 5 \\ -6 & 100 & 5 \\ 5 & 5 & 100 \end{bmatrix} \text{MPa.}$$

Assume $\tau_Y = 20$ MPa.

(9.26) Solve Exercise 9.25 using the Henky–von Mises condition with $\sigma_Y = 30$ MPa.

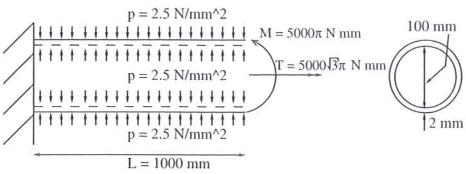
(9.27) What is the pressure (mean normal stress) for the following state of stresses:

$$\begin{bmatrix} 30 & -6 & 8 \\ -6 & 10 & 15 \\ 8 & 15 & 10 \end{bmatrix} \text{ksi.}$$

What is the corresponding deviatoric stress for this state of stress?

(9.28) Consider a solid round bar of radius R and length L. Compute the total elastic energy in the bar when it is subjected to an axial end-load P. Compute the total deviatoric energy in the bar. Compute the total hydrostatic/volumetric energy in the bar. Do your last two expressions add up to the first?

(9.29) Consider the thin-walled tube as shown. The tube is subjected to a bending moment and a torque. The inner and outer walls are also subject to a (uniform) pressure. The tube has a yield stress in shear $\tau_Y = 1.25\ \text{N/mm}^2$. Apply *Tresca's* condition and determine whether or not the tube will yield under the given loads.



(9.30) Solve Exercise 9.29 using the Henky–von Mises conditions. Assume $\sigma_Y = 1.25\sqrt{3}\ \text{N/mm}^2$.

(9.31) At what internal pressure does a spherical pressure vessel yield according to Tresca's condition?

(9.32) Consider a cantilevered solid round bar of length L and radius R. Determine the allowable combinations of applied end-moment M and end-torque T according to the Henky–von Mises condition. Make a plot of the elastic domain in the M-T plane, and clearly label all important points defining the elastic domain.

2L (2x)

11 Potential-Energy Methods

In this chapter we will examine the use of potential-energy methods to solve problems similar to those we looked at in Chapter 10. In fact, we will see in the second part of this chapter that the methods we will develop will lead to equations that are strikingly similar to those used in the method of virtual forces. This similarity is not accidental as potential energy methods are intimately related virtual work methods. Thus the techniques we will use in this chapter can be used as an alternate way of understanding virtual work. It is, however, noted that the concepts of virtual work are more general than potential energy methods – potential-energy methods apply only to conservative systems, while virtual work methods can be applied to conservative and non-conservative systems.

11.1 Potential energy: Spring-mass system

As an introduction to potential-energy methods let us recall the familiar example of a mass with weight P resting on top of a linear spring with spring constant k, as shown in Fig. 11.1. Before we place the mass on the spring, let us assume that it has a length z_o. After we place the mass on the spring, the spring will compress an amount Δ and the mass will come to a static equilibrium position $z = z_o - \Delta$. We know for static equilibrium of the mass that the sum of the gravitational force and the spring force (sum of the forces in the z-direction) must be zero:

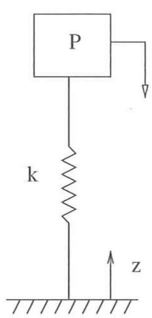


Fig. 11.1 Spring-mass system.

$$k\Delta - P = 0. \tag{11.1}$$

Both of the forces acting on the mass happen to be conservative. From Chapter 1, we know that the gravitational force can be expressed as the potential

$$\Pi_{\text{gravity}} = Pz = P(z_o - \Delta), \tag{11.2}$$

where gravity is acting downwards and thus the gravity force acting on the mass is $F_{\text{gravity}} = -d\Pi/dz = -P$. The spring force itself is $F_{\text{spring}} = k\Delta = k(z_o - z)$; i.e. for positive motion Δ the spring pushes up on the mass. If we want, we can also express this force in terms of the potential

$$\Pi_{\text{spring}} = \frac{1}{2}k(z_o - z)^2 = \frac{1}{2}k\Delta^2 \tag{11.3}$$

(period)

Remarks:

(1) From the result we can conclude with some confidence, for example, that with only three terms our result is accurate to three digits or has error less than 0.1%. Also shown in Table 11.1 is the exact solution computed by solving the governing differential equation. This verifies our conclusion.

(2) This type of analysis is not an exact error analysis but suffices in most situations.

(3) If we had not restricted i to be odd, we would have had essentially the same result. The main difference would have been that half of the generalized displacements would have been zero.

11.9.3 Selecting functions for Ritz's method

An important aspect of selecting the functions for Ritz's method, beyond the requirement that they satisfy the kinematic boundary conditions, is that with increasing N the new functions add to the approximation space without overlapping too much with the other functions. Mathematically this is expressed by trying to use functions that are as *orthogonal* to each other as possible. With ordinary vectors, say $\boldsymbol{a} = a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2$ and $\boldsymbol{b} = b_1\boldsymbol{e}_1 + b_2\boldsymbol{e}_2$, we define orthogonality as the requirement that their inner (or dot) product be zero:

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle = a_1 b_1 + a_2 b_2 = \|\boldsymbol{a}\| \, \|\boldsymbol{b}\| \cos(\theta_{ab}) = 0, \qquad (11.122)$$

where we have utilized the corresponding norm for such vectors $\|\boldsymbol{a}\| = \sqrt{\langle \boldsymbol{a}, \boldsymbol{a} \rangle}$, and θ_{ab} is the angle between the two vectors. When we are measuring error with the L^2 norm we also have a corresponding L^2 inner product between functions. If $f(x)$ and $g(x)$ are two functions, then their L^2 inner product is given as

$$\langle f(x), g(x) \rangle = \sqrt{\int_0^L f(x) g(x) \, dx}. \qquad (11.123)$$

remove square root symbol

The abstract angle between two functions is defined via

$$\cos(\theta_{fg}) = \frac{\langle f(x), g(x) \rangle}{\|f(x)\| \, \|g(x)\|}. \qquad (11.124)$$

So orthogonality between functions occurs when their inner product is zero – just as with ordinary vectors.

Remarks:

(1) In certain problems it is convenient to choose a set of orthogonal functions but in others it is not. Even if full orthogonality can

$$M_z = \int_A -\sigma_{xx} y \, dA \,, \tag{F.5}$$

$$M_y = \int_A \sigma_{xx} z \, dA \,. \tag{F.6}$$

To understand the issue of twisting we need to consider the shear stresses on the cross-section in detail. When we bend a beam with transverse forces we will directly generate shear stresses, say, σ_{xy}^{DS} and σ_{xz}^{DS}; i.e. direct shear stresses. If we separately twist the beam we will also have contributions to the shear stresses due to the applied twist, say σ_{xy}^{AT} and σ_{xz}^{AT}. The total shear stresses on the cross-section, in general, will be the sum of these two contributions.

The torque associated with the shear stresses from the applied twist is given by

$$T^{AT} = \int_A -\sigma_{xy}^{AT} z + \sigma_{xz}^{AT} y \, dA \,. \tag{F.7}$$

Since we are assuming that the system is linear elastic, we can also assume that this torque is related to the applied twist rate as:

$$T^{AT} = C \frac{d\phi}{dx} \,, \tag{F.8}$$

where C represents the effective torsional stiffness of the cross-section – i.e. a sort of $(GJ)_{\text{eff}}$. Thus ~~the~~ we can express the total torque on the cross-section as

$$T = C \frac{d\phi}{dx} + \int_A -\sigma_{xy}^{DS} z + \sigma_{xz}^{DS} y \, dA \,. \tag{F.9}$$

If it is desired that $d\phi/dx = 0$ (i.e. the beam does not twist), then there must be a net torque on the cross-section (about the centroidal axis) of

$$T_{\text{no twist}} = \int_A -\sigma_{xy}^{DS} z + \sigma_{xz}^{DS} y \, dA \,. \tag{F.10}$$

Remarks:

(1) This final result states that if we wish to bend a beam with shear forces through the centroid of a cross-section without it simultaneously twisting, then in general we will need to also apply a torque about the the centroidal axis of the beam. For circular and rectangular cross-sections this torque turns out to be zero. However, for more general shapes it is often non-zero.

(2) One common way of generating this torque is to shift the applied shear forces away from the centroid, as we will see in the example below.

(3) Note that for the argument presented, one in principle needs to know how to compute the torsional stiffness of a general cross-section. However, in the end, we want to enforce the desire that the twist is zero, and thus we never actually need to be able to compute the constant C above.