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University of California, Berkeley

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R. L. Taylor
Department of Civil and Environmental Engineering

Univeristy of California, Berkeley

and

P. Wriggers
Institut für Baumechanik und Numerische Mechanik

Universität Hannover, Germany

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1 Introduction

Finite deformation contact problems are associated with large sliding in the contact area. Thus, during an analysis a slave node can slide over several master segments within the actual finite element discretization.

The standard contact discretization in these cases is based a master-slave formulation in which the slave node contacts a straight master segment. Due to this treatment the normal changes from segment to segment without a smooth transition when passing from one segment to the next. Since this may lead to convergence problems and furthermore may initiate jumps in the velocity field in dynamic solutions, it is preferable to have a smooth contact discretization with a continuous normal field. To have a continuous surface with no slope discontinuities between segments, a C^1–continuous interpolation of the master surface is necessary. To achieve this, one can use different forms of discretizations. Among these are Bezier, Hermitian or other types of spline interpolations. In this paper we compare two formulations and discretizations which can be used to obtain smooth normal fields for contact of two deformable bodies.

Work regarding smooth contact has first been devoted for finite deformations to the contact of deformable bodies with rigid surfaces, see e.g. [1] or [2] or [3]. For the contact of two or more deformable bodies see [4] or [5].

2 Formulation of frictionless contact problems

We consider two elastic bodies $\mathcal{B}^\alpha, \alpha = 1, 2$, each occupying a bounded domain $\Omega^\alpha \subset R^2$, that means we restrict ourselves in this paper to two dimensional contact problems. The

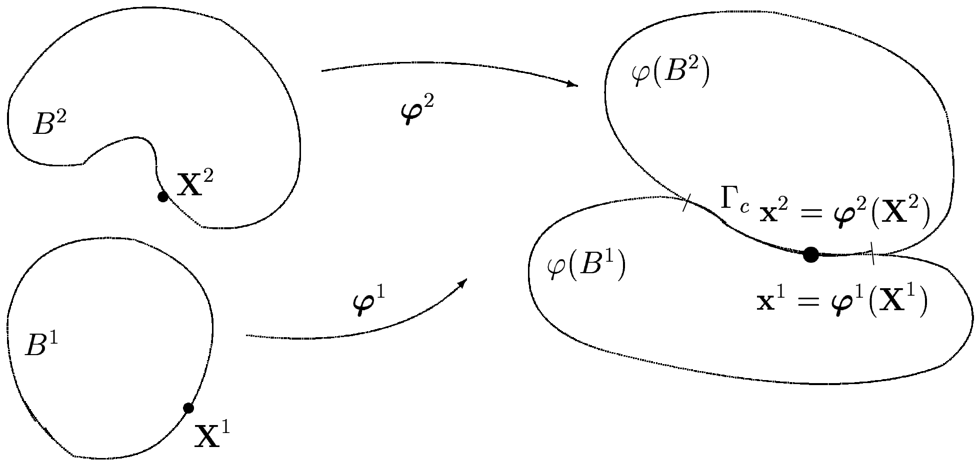

Figure 1: Large deformations and contact of two bodies.

mapping $\boldsymbol{\varphi}^\alpha$ maps points in the initial configuration, described by the position vector $\mathbf{X}^\alpha$ to points in the deformed configuration $\mathbf{x}^\alpha = \boldsymbol{\varphi}^\alpha(\mathbf{X}^\alpha)$, see Figure 1.

The boundary Γ^α of a body $\mathcal{B}^\alpha$ consists of three parts: Γ^α_σ with prescribed surface loads, Γ^α_u with prescribed displacements and Γ^α_c where the two bodies $\mathcal{B}^1$ and $\mathcal{B}^2$ come into contact. In the contact area we have to formulate the constraint equations for the normal contact as well as the kinematical relations for the tangential contact. Here on the deformed contact surface Γ_c two distinct points in the initial configuration, $\mathbf{X}^1$ and $\mathbf{X}^2$, will have the same position $\mathbf{x}^1 = \mathbf{x}^2$, see Figure 1.

Furthermore we will state the weak form for bodies undergoing large elastic deformations which leads in case of contact to a variational inequality.

2.1 Contact kinematics

Since the deformation of two bodies in space can be arbitrary (see Figure 1) and may consist of finite rotations and large deflections a global search procedure is needed to find parts of the bodies, denoted by Γ_c which come into contact. This search process can be based on methods like bucket search or binary tree algorithms, but it will not be discussed here in detail.

Once the contact surface Γ_c is known, the local geometrical contact conditions can be established. For this purpose we describe the surface on the bodies by convective coordinates. Furthermore, we define one the surface of one of the contacting bodies, $\mathcal{B}^1$, as the *master* surface which means that this surface is the reference surface for the subsequent derivations. The other surface of body $\mathcal{B}^2$ is called the *slave* surface. This choice is arbitrary and completely interchangeable since in the final solution of the contact problem all geometrical quantities of both surfaces (e.g., the normal vectors) coincide.

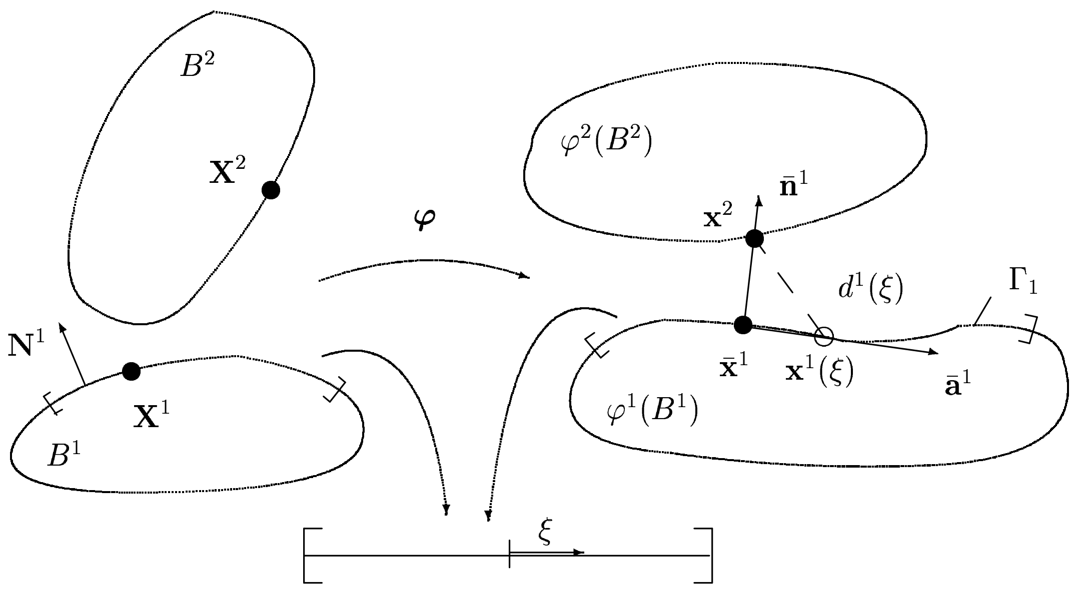

Figure 2: Definition of the gap in finite deformations.

Figure 2 shows the parameterization of the contact surface for the two–dimensional case. $\mathbf{x}^\alpha = \boldsymbol{\varphi}^\alpha(\mathbf{X}^\alpha)$ denotes the coordinates of the current configuration of body $\mathcal{B}^\alpha$: $\mathbf{x}^\alpha = \mathbf{X}^\alpha + \mathbf{u}^\alpha$ where $\mathbf{X}^\alpha$ is related to the reference configuration and $\mathbf{u}^\alpha$ is the displacement field. The normal vector $\mathbf{n}_1$ is associated with the *master* body $\mathcal{B}^1$. Assuming that the contact boundary describes, at least locally, a convex region we can relate to every point $\mathbf{x}^2$ on Γ^2 a point $\bar{\mathbf{x}}^1 = \mathbf{x}^1(\bar{\xi})$ on Γ^1 via the minimal distance problem, see Figure 2,

$$\|\mathbf{x}^2 - \bar{\mathbf{x}}^1\| = \min\, d^1(\xi) = \min_{\mathbf{x}^1 \subseteq \Gamma^1} \|\mathbf{x}^2 - \mathbf{x}^1(\xi)\|\,, \tag{1}$$

where ξ denotes the parameterization of the boundary Γ^1, see e.g. [6]. The minimization process yields the condition $(\mathbf{x}^2 - \bar{\mathbf{x}}^1) \cdot \bar{\mathbf{a}}^1 = 0$ which means that $(\mathbf{x}^2 - \bar{\mathbf{x}}^1)$ points in the direction of $\bar{\mathbf{n}}^1$, see Figure 2. We will use in the following the notation $\bar{\mathbf{x}}^1 = \mathbf{x}^1(\xi)|_{\xi=\bar{\xi}}$.

Once the point $\bar{\mathbf{x}}^1$ is known, we can write the geometrical contact constraint inequality which prevents penetration of one body into the other

$$\bar{g}_N = (\mathbf{x}^2 - \bar{\mathbf{x}}^1) \cdot \bar{\mathbf{n}}^1 \geq 0 \tag{2}$$

In view of the penalty formulation which will be applied here to solve the contact problems we introduce a penetration function to allow for a small penetrations on Γ_c:

$$g_N = \begin{cases} (\mathbf{x}^2 - \bar{\mathbf{x}}^1) \cdot \bar{\mathbf{n}}^1 & \text{if } (\mathbf{x}^2 - \bar{\mathbf{x}}^1) \cdot \bar{\mathbf{n}}^1 < 0 \\ 0 & \text{otherwise}\,. \end{cases} \tag{3}$$

Function g_N indicates a penetration of one body into the other and shows in which parts of Γ^α the constraint equations preventing penetration have to be activated. Thus (3) can be used to determine the contact area $\Gamma_c^\alpha \subseteq \Gamma^\alpha$.

Remark I: *In the case of contact between a rigid surface and a deformable body equation (3) also holds. Then* $\bar{\mathbf{x}}^1$ *are the coordinates and* $\bar{\mathbf{n}}^1$ *is the normal of the rigid body.*

Remark II: *Due to the closest point projection we can write*

$$\mathbf{x}^2 - \bar{\mathbf{x}}^1 = g_N \, \bar{\mathbf{n}}^1 \,. \tag{4}$$

2.2 Weak form for solids in finite elasticity

Since the weak form for elastic bodies is quite standard, we only want to discuss here the contact contributions. As introduced above the contact conditions are given by the geometrical non–penetration condition (2)

$$g_N \geq 0 \tag{5}$$

Furthermore we do not consider adhesion on the contact interface thus tension stresses cannot occur on the contact area, thus the normal traction inequality is given as

$$p_N \leq 0 \,. \tag{6}$$

For a numerical solution of the nonlinear boundary value problem associated with finite elasticity we will use the finite element method. If W^α is the strain energy function of the elastic bodies $\mathcal{B}^\alpha$ then we can write the total energy as

$$\Pi = \sum_{\alpha=1}^{2} \int\limits_{\Omega^\alpha} W^\alpha \, dV - \int\limits_{\Omega^\alpha} \bar{\mathbf{f}}^\alpha \cdot \boldsymbol{\varphi}^\alpha \, dV \int\limits_{\partial\Omega^\alpha} \bar{\mathbf{t}}^\alpha \cdot \boldsymbol{\varphi}^\alpha \, dA \tag{7}$$

There are several methods to solve (7) when contact constraints are present. A common approach is an active set strategy combined with the penalty method, see e.g. ODEN [7] or KIKUCHI, ODEN [8], which also is employed in this paper. Introducing a penalty constraint for the normal contact leads, by assuming that the contact surface is known, instead of (7) to the minimization problem

$$\Pi + \Pi_\varepsilon \Rightarrow MIN \tag{8}$$

where the part regarding the contact constraints is given by

$$\Pi_\varepsilon = \int\limits_{\Gamma_c} \varepsilon \, g_N^2 \, d\Gamma \qquad \varepsilon > 0 \tag{9}$$

and ε denotes the penalty parameter. In the discretization that we will use, the contact constraint is enforced for each slave node. Hence the integral in (9) can be written as a sum

$$\Pi_\varepsilon^h = \sum_{s=1}^{n_c} \frac{\varepsilon}{2} \, g_{N_s}^2 \, A_s \tag{10}$$

Here n_c is the number of active constraints and A_s is an area related to the contact node s.

The minimum of (8) is found when the first variation is zero. Since we are mainly interested in the contact formulation, we omit in the following the contribution Π related to the solids.

2.2.1 Form A

The first variation of Π_ε is

$$\mathrm{d}\,\Pi_\varepsilon^h = \sum_{s=1}^{n_c} \varepsilon\, \mathrm{d}g_{N_s}\, g_{N_s}\, A_s \tag{11}$$

Within this formulation we need to compute the variation δg_{N_s} which follows from (4) as

$$\mathrm{d}g_{N_s}\, \bar{\mathbf{n}}^1 + g_{N_s}\, \mathrm{d}\,\bar{\mathbf{n}}^1 = \delta \mathbf{x}_s^2 - \delta \bar{\mathbf{x}}^1 - \bar{\mathbf{x}}^1_{,\xi}\, \delta\xi \tag{12}$$

where the last term denotes the variation of the surface coordinate ξ. The notation is chosen such that $\mathrm{d}g(\mathbf{x}(\xi),\xi) = (\partial g \,/\, \partial \mathbf{x})\, \delta\mathbf{x} + \partial g \,/\, \partial\xi\, \delta\xi$ denotes the total variation whereas $\delta g(\mathbf{x}(\xi),\xi) = (\partial g \,/\, \partial \mathbf{x})\, \delta\mathbf{x}$ is the variation with respect to the variable $(\mathbf{x})$.

Taking the scalar product with $\bar{\mathbf{n}}^1$ yields the final result for the variation of the normal gap

$$\mathrm{d}g_{N_s} = (\delta \mathbf{x}_s^2 - \delta \bar{\mathbf{x}}^1) \cdot \bar{\mathbf{n}}^1 \tag{13}$$

Since we want to use Newton's method to solve the nonlinear equations the linearization of the variation $\mathrm{d}\Pi_\varepsilon$ is necessary. To obtain a symmetric tangent matrix we start from (11) and get

$$\mathrm{Dd}\Pi_\varepsilon^h = \sum_{s=1}^{n_c} \varepsilon\, \mathrm{d}g_{N_s}\, \mathrm{D}g_{N_s}\, A_s + \varepsilon\, g_{N_s}\, \mathrm{Dd}g_{N_s}\, A_s \tag{14}$$

The linearization of the gap, $\mathrm{D}g_{N_s}$, in the first term has the same structure as $\mathrm{d}g_{N_s}$, thus we have

$$\mathrm{D}g_{N_s} = (\Delta \mathbf{x}_s^2 - \Delta \bar{\mathbf{x}}^1) \cdot \bar{\mathbf{n}}^1 \tag{15}$$

Again D stands for the total linearization and Δ for the linearization with respect to the variable $\mathbf{x}$. The term $\mathrm{Dd}g_{N_s}$ is obtained from the linearization of (12)

$$\mathrm{Dd}g_{N_s}\, \bar{\mathbf{n}}^1 + \mathrm{d}g_{N_s}\, \mathrm{D}\bar{\mathbf{n}}^1 + \mathrm{D}g_{N_s}\, \mathrm{d}\bar{\mathbf{n}}^1 + g_{N_s}\, \mathrm{Dd}\bar{\mathbf{n}}^1 = -\delta\bar{\mathbf{x}}^1_{,\xi}\, \Delta\xi - \Delta\bar{\mathbf{x}}^1_{,\xi}\, \delta\xi - \bar{\mathbf{x}}^1_{,\xi\xi}\, \delta\xi\, \Delta\xi - \Delta\delta\bar{\mathbf{x}}^1 \tag{16}$$

Noting that $\bar{\mathbf{n}}^1 \cdot \mathrm{D}\bar{\mathbf{n}}^1 = \bar{\mathbf{n}}^1 \cdot \mathrm{d}\bar{\mathbf{n}}^1 = 0$, the scalar product with $\bar{\mathbf{n}}^1$ yields

$$\mathrm{Dd}g_{N_s} = -g_{N_s}\, \mathrm{Dd}\bar{\mathbf{n}}^1 \cdot \bar{\mathbf{n}}^1 - \delta\bar{\mathbf{x}}^1_{,\xi} \cdot \bar{\mathbf{n}}^1\, \Delta\xi - \Delta\bar{\mathbf{x}}^1_{,\xi} \cdot \bar{\mathbf{n}}^1\, \delta\xi - \bar{\mathbf{x}}^1_{,\xi\xi} \cdot \bar{\mathbf{n}}^1\, \delta\xi\, \Delta\xi - \Delta\delta\bar{\mathbf{x}}^1 \cdot \bar{\mathbf{n}}^1 \tag{17}$$

where the first term on the right hand side can be rewritten, since

$$\mathrm{D}[\,\mathrm{d}\bar{\mathbf{n}}^1 \cdot \bar{\mathbf{n}}^1\,] = \mathrm{Dd}\bar{\mathbf{n}}^1 \cdot \bar{\mathbf{n}}^1 + \mathrm{d}\bar{\mathbf{n}}^1 \cdot \mathrm{D}\bar{\mathbf{n}}^1 = 0 \Rightarrow \mathrm{D}\,\mathrm{d}\bar{\mathbf{n}}^1 \cdot \bar{\mathbf{n}}^1 = -\mathrm{d}\bar{\mathbf{n}}^1 \cdot \mathrm{D}\bar{\mathbf{n}}^1$$

So finally we have

$$\mathrm{Dd}g_{N_s} = g_{N_s}\, \mathrm{d}\bar{\mathbf{n}}^1 \cdot \mathrm{D}\bar{\mathbf{n}}^1 - \delta\bar{\mathbf{x}}^1_{,\xi} \cdot \bar{\mathbf{n}}^1\, \Delta\xi - \Delta\bar{\mathbf{x}}^1_{,\xi} \cdot \bar{\mathbf{n}}^1\, \delta\xi - \bar{\mathbf{x}}^1_{,\xi\xi} \cdot \bar{\mathbf{n}}^1\, \delta\xi\, \Delta\xi - \Delta\delta\bar{\mathbf{x}}^1 \cdot \bar{\mathbf{n}}^1 \tag{18}$$

In this formulation we now have to compute the variation and linearization of the normal $\bar{\mathbf{n}}^1$ and the surface coordinate ξ and express these in terms of the variables $\mathbf{x}_s^2$ and $\bar{\mathbf{x}}^1$.

Since $\bar{\mathbf{n}}^1 \cdot \bar{\mathbf{x}}^1_{,\xi} = 0$ the variation of this product yields

$$\mathrm{d}\bar{\mathbf{n}}^1 \cdot \bar{\mathbf{x}}^1_{,\xi} + \bar{\mathbf{n}}^1 \cdot \mathrm{d}\bar{\mathbf{x}}^1_{,\xi} = 0$$

With the definition of the unit tensor

$$\mathbf{1} = \left[\bar{\mathbf{n}}^\mathbf{1} \otimes \bar{\mathbf{n}}^\mathbf{1} + \frac{\mathbf{1}}{\| \bar{\mathbf{x}}^\mathbf{1}_{,\xi} \|^\mathbf{2}} \bar{\mathbf{x}}^\mathbf{1}_{,\xi} \otimes \bar{\mathbf{x}}^\mathbf{1}_{,\xi} \right]$$

we obtain from (2.2.1) after some algebra

$$\mathrm{d}\bar{\mathbf{n}}^1 = -\frac{1}{\| \bar{\mathbf{x}}^1_{,\xi} \|^2} \left[\bar{\mathbf{x}}^1_{,\xi} \otimes \bar{\mathbf{n}}^1 \right] \mathrm{d}\bar{\mathbf{x}}^1_{,\xi} \, . \tag{19}$$

Note that in this expression the variation of $\bar{\mathbf{x}}^1$ involves also the change of the surface coordinate

$$\mathrm{d}\bar{\mathbf{x}}^1 = \delta \bar{\mathbf{x}}^1 + \bar{\mathbf{x}}^1_{,\xi} \, \delta \xi \, . \tag{20}$$

Since however $\bar{\mathbf{x}}^1_{,\xi} \cdot \bar{\mathbf{n}}^1 = 0$ we can neglect the second term in (19).

The variation of the surface coordinate, needed in (18), can be computed from the change of gap in tangent direction, $(\mathbf{x}^2_s - \bar{\mathbf{x}}^1) \cdot \bar{\mathbf{x}}^1_{,\xi} = 0$, which yields after some manipulations

$$\delta \xi = \frac{1}{\bar{\mathbf{x}}^1_{,\xi} \cdot \bar{\mathbf{x}}^1_{,\xi} - g_{N_s} \, \bar{\mathbf{n}}^1 \cdot \bar{\mathbf{x}}^1_{,\xi\xi}} \left[(\delta \mathbf{x}^2_s - \delta \bar{\mathbf{x}}^1) \cdot \bar{\mathbf{x}}^1_{,\xi} + g_{N_s} \, \bar{\mathbf{n}}^1 \cdot \delta \bar{\mathbf{x}}^1_{,\xi} \right] \tag{21}$$

By using the sam formalism to derive $\Delta \xi$ and by inserting (21) and (19) into (18) we obtain the final expression for the linearization of the gap variation which is given in terms of the primary variables $\mathbf{x}^2_s$ and $\bar{\mathbf{x}}^1$.

Before proceeding to the smooth contact discretization we investigate a different derivation of the penalty formulation for the contact constraints.

2.2.2 Form B

Note that the penalty term (10) can also be written with (4) as

$$\begin{aligned} \Pi^h_\varepsilon &= \sum_{s=1}^{n_c} \frac{\varepsilon}{2} g_{N_s} \bar{\mathbf{n}}^1 \cdot g_{N_s} \bar{\mathbf{n}}^1 \, A_s \\ &= \sum_{s=1}^{n_c} \frac{\varepsilon}{2} (\mathbf{x}^2_s - \bar{\mathbf{x}}^1) \cdot (\mathbf{x}^2_s - \bar{\mathbf{x}}^1) \, A_s \end{aligned} \tag{22}$$

where now the normal no longer appears. The variation of this expression yields

$$\mathrm{d}\Pi^h_\varepsilon = \sum_{s=1}^{n_c} \varepsilon \, (\delta \mathbf{x}^2_s - \delta \bar{\mathbf{x}}^1 - \bar{\mathbf{x}}^1_{,\xi} \, \delta \xi) \cdot (\mathbf{x}^2_s - \bar{\mathbf{x}}^1) \, A_s \tag{23}$$

Again, since $(\mathbf{x}^2_s - \bar{\mathbf{x}}^1) = g_{N_s} \, \bar{\mathbf{n}}^1$ the third term in the first bracket disappears and we have for the variation of Π_ε

$$\mathrm{d}\Pi^h_\varepsilon = \sum_{s=1}^{n_c} \varepsilon \, (\delta \mathbf{x}^2_s - \delta \bar{\mathbf{x}}^1) \cdot (\mathbf{x}^2_s - \bar{\mathbf{x}}^1) \, A_s \tag{24}$$

The linearization $\mathrm{Dd}\Pi^h_\varepsilon$ starts from (23) and leads to the symmetric form

$$\begin{aligned}\mathrm{Dd}\Pi^h_\varepsilon &= \sum_{s=1}^{n_c} \varepsilon\, A_s \left[(\delta \mathbf{x}^2_s - \delta \bar{\mathbf{x}}^1 - \bar{\mathbf{x}}^1_{,\xi}\, \delta\xi) \cdot (\Delta \mathbf{x}^2_s - \Delta \bar{\mathbf{x}}^1 - \bar{\mathbf{x}}^1_{,\xi}\, \Delta\xi) \right. \\ &+ \left. (\mathbf{x}^2_s - \bar{\mathbf{x}}^1) \cdot (-\Delta\delta\bar{\mathbf{x}}^1 - \delta\bar{\mathbf{x}}^1_{,\xi}\, \Delta\xi - \Delta\bar{\mathbf{x}}^1_{,\xi}\, \delta\xi - \bar{\mathbf{x}}^1_{,\xi\xi} \cdot \bar{\mathbf{n}}^1\, \delta\xi\Delta\xi\,) \right] \end{aligned} \tag{25}$$

This expression can be put into a mixed form by using vector notation. After collecting terms we arrive at

$$\mathrm{Dd}\Pi^h_\varepsilon = \sum_{s=1}^{n_c} \varepsilon\, A_s [\langle\, \delta\mathbf{x}^2_s \,, \delta\bar{\mathbf{x}}^1 \,, \delta\xi \,\rangle \begin{bmatrix} \mathbf{I} & -\mathbf{I} & -\bar{\mathbf{x}}^1_{,\xi} \\ -\mathbf{I} & \mathbf{I} & \bar{\mathbf{z}}^1 \\ -\bar{\mathbf{x}}^{1\,T}_{,\xi} & \bar{\mathbf{z}}^{1\,T} & H_{\xi\xi} \end{bmatrix} \begin{Bmatrix} \Delta\mathbf{x}^2_s \\ \Delta\bar{\mathbf{x}}^1 \\ \Delta\xi \end{Bmatrix} - (\mathbf{x}^2_s - \bar{\mathbf{x}}^1)^T\, \Delta\delta\bar{\mathbf{x}}^1] \tag{26}$$

where the abbreviations

$$\begin{aligned} H_{\xi\xi} &= (\bar{\mathbf{x}}^1_{,\xi})^T\, \bar{\mathbf{x}}^1_{,\xi} - g_{N_s}\, (\bar{\mathbf{n}}^1)^T\, \bar{\mathbf{x}}^1_{,\xi\xi} \\ \bar{\mathbf{z}}^1 &= \bar{\mathbf{x}}^1_{,\xi} - g_{N_s}\, \bar{\mathbf{n}}^1 \end{aligned}$$

have been used. Since the variation (24) does not depend on $\delta\xi$ we can use a static condensation to eliminate $\Delta\xi$ in (26). This yields

$$\mathrm{Dd}\Pi^h_\varepsilon = \sum_{s=1}^{n_c} \varepsilon\, A_s [\langle\, \delta\mathbf{x}^2_s \,, \delta\bar{\mathbf{x}}^1 \,\rangle\, \mathbf{A}(\bar{\xi}) \begin{Bmatrix} \Delta\mathbf{x}^2_s \\ \Delta\bar{\mathbf{x}}^1 \end{Bmatrix} - (\mathbf{x}^2_s - \bar{\mathbf{x}}^1)^T\, \Delta\delta\bar{\mathbf{x}}^1] \tag{27}$$

where the matrix

$$\mathbf{A}(\bar{\xi}) = \begin{bmatrix} \mathbf{I} - H^{-1}_{\xi\xi}\, \bar{\mathbf{x}}^1_{,\xi}\, \bar{\mathbf{x}}^{1\,T}_{,\xi} & -\mathbf{I} + H^{-1}_{\xi\xi}\, \bar{\mathbf{x}}^1_{,\xi}\, \bar{\mathbf{z}}^{1\,T} \\ -\mathbf{I} + H^{-1}_{\xi\xi}\, \bar{\mathbf{z}}^1\, \bar{\mathbf{x}}^{1\,T}_{,\xi} & \mathbf{I} - H^{-1}_{\xi\xi}\, \bar{\mathbf{z}}^1\, \bar{\mathbf{z}}^{1\,T} \end{bmatrix} \tag{28}$$

has been introduced.

Note that we recover in this elimination process also the formula for the variation of the surface coordinate (21).

Now we have to discretize the master surface of body $\mathcal{B}^1$ which then yields the matrix form for the smooth contact element.

3 Smooth contact discretization

Here we discuss two different interpolations which yield a C^1 continuous surface interpolation of the master surface. The first is a Hermitian interpolation and the second is a Bezier interpolation. These two interpolations lead to different results as discussed later. There are many more possibilities to derive C^1 continuous contact interpolations, like splines or B–splines. However these are not considered here.

An interpolation which provides a normal field which does not have jumps when going from one segment to the next has to be C^1–continuous. There are different possibilities to set up the interpolation. The first one investigated is the Hermitian interpolation, then we will present the matrix formulation for a C^1–continuous Bezier interpolation.

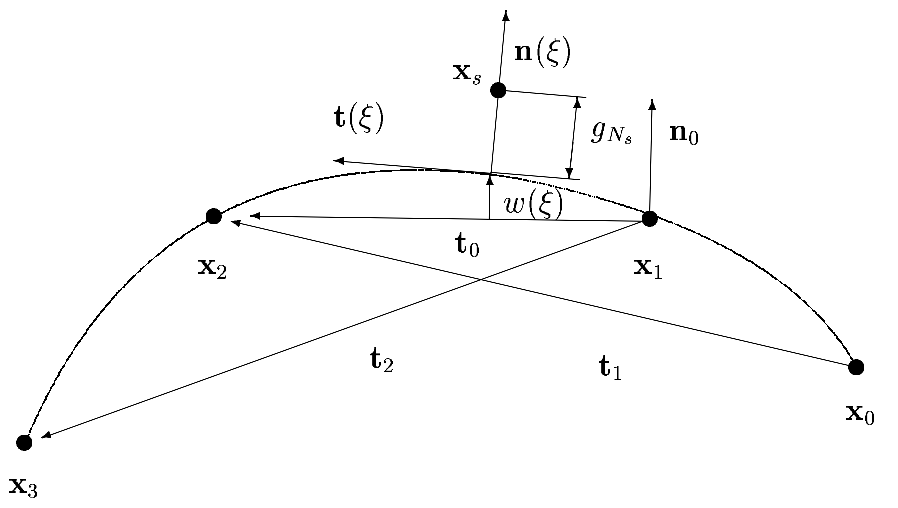

Figure 3: C^1 – continuous interpolation of contact surface.

3.1 Smooth Hermitian interpolation

Since Hermitian polynomials are cubic functions we need four nodal points on the master surface, see Figure 3, to define the interpolation.

To obtain continuous tangents from one segment to the next – the one under consideration here is denoted by the nodes $\mathbf{x}_1$ and $\mathbf{x}_2$ – we define the tangent vectors $\mathbf{t}_1 = \mathbf{x}_2 - \mathbf{x}_0$ and $\mathbf{t}_2 = \mathbf{x}_3 - \mathbf{x}_1$.

Let us furthermore introduce a tangent and a normal vector which form a local frame for the segment under consideration

$$\mathbf{t}_0 = \mathbf{x}_2 - \mathbf{x}_1 \qquad \text{and} \quad \mathbf{n}_0 = -\mathbf{e}_3 \times \mathbf{t}_0 = \mathbf{T}\,\mathbf{t}_0 \tag{29}$$

where $\mathbf{e}_3$ is the unit base vector perpendicular to the plane, thus the cross product can be expressed by the skew matrix $\mathbf{T}$

$$\mathbf{T} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{30}$$

Now we can define the surface interpolation as a linear interpolation between nodes 1 and 2 and a cubic Hermitian interpolations with respect to the local frame

$$\mathbf{x}(\xi) = N_1(\xi)\,\mathbf{x}_1 + N_2(\xi)\,\mathbf{x}_2 + w(\xi)\,\mathbf{n}_0 \tag{31}$$

where the cubic interpolation is given by $w(\xi) = H_1(\xi)\,B_1 + H_2(\xi)\,B_2$. $N_\alpha(\xi)$ being the standard linear shape functions and $H_\alpha(\xi)$ being the Hermitian polynomials defined as

$$\begin{aligned} N_1(\xi) &= \frac{1}{2}(1-\xi) \\ N_2(\xi) &= \frac{1}{2}(1+\xi) \\ H_1(\xi) &= \frac{1}{4}(\xi^2-1)(\xi-1) \\ H_2(\xi) &= \frac{1}{4}(\xi^2-1)(\xi+1) \end{aligned} \tag{32}$$

The angle B_α is given in terms of the tangent vectors $\mathbf{t}_\alpha$ as

$$B_\alpha = \frac{1}{2}\,\frac{\mathbf{t}_\alpha^T\,\mathbf{n}_0}{\mathbf{t}_\alpha^T\,\mathbf{t}_0} \tag{33}$$

and denotes the angle between the tangent $\mathbf{t}_\alpha$ and the local frame defined by $(\mathbf{t}_0\,,\mathbf{n}_0)$. With this we can summarize the interpolation of the surface within the segment between node 1 and 2 as

$$\mathbf{x}(\xi) = \sum_{\alpha=1}^{2}\left[\,N_\alpha(\xi)\,\mathbf{x}_\alpha + H_\alpha(\xi)\,B_\alpha\,\mathbf{n}_0\right] \tag{34}$$

Now we have to express the variation and the linearization of the gap associated with one slave node $\mathbf{x}_s$. For this, the derivative of $\mathbf{x}(\xi)$ with respect to the surface coordinate ξ is also needed and hence given next

$$\mathbf{x}(\xi)_{\,,\xi} = \frac{1}{2}(\mathbf{x}_2 - \mathbf{x}_1) + \sum_{\alpha=1}^{2} H_{\alpha\,,\xi}(\xi)\,B_\alpha\,\mathbf{n}_0 \tag{35}$$

The variation of $\mathbf{x}(\xi)$ yields

$$\delta\mathbf{x}(\xi) = \sum_{\alpha=1}^{2}\left[\,N_\alpha(\xi)\delta\mathbf{x}_\alpha + H_\alpha(\xi)\,\left(\,\delta B_\alpha\,\mathbf{n}_0 + B_\alpha\,\delta\mathbf{n}_0\right)\right] \tag{36}$$

By exploiting the structure of $\delta B_\alpha = \frac{1}{2}\,\delta\left(\frac{\mathbf{t}_\alpha^T\,\mathbf{n}_0}{\mathbf{t}_\alpha^T\,\mathbf{t}_0}\right)$ we arrive at the final form for δB_α

$$\delta B_\alpha = \frac{1}{2\,\mathbf{t}_\alpha^T\mathbf{t}_0}\,\left[\,(\mathbf{t}_\alpha^T\,\mathbf{T}^T - B_\alpha\,\mathbf{t}_\alpha^T\,)\,\delta\mathbf{t}_0 + (\mathbf{n}_0^T - B_\alpha\,\mathbf{t}_0^T\,)\,\delta\mathbf{t}_\alpha\,\right] \tag{37}$$

Now we can write the variation of B_α in matrix form. For this we define

$$\begin{aligned} \mathbf{p}_1^{(\alpha)} &= \frac{H_\alpha}{2\,\mathbf{t}_\alpha^T\,\mathbf{t}_0}\left[\,\mathbf{T}\,\mathbf{t}_\alpha - B_\alpha\,\mathbf{t}_\alpha\,\right] \\ \mathbf{p}_2^{(\alpha)} &= \frac{H_\alpha}{2\,\mathbf{t}_\alpha^T\,\mathbf{t}_0}\left[\,\mathbf{n}_0 - B_\alpha\,\mathbf{t}_0\,\right] \end{aligned} \tag{38}$$

and obtain

$$\delta B_\alpha = \langle\, \delta \mathbf{t}_0 \,,\, \delta \mathbf{t}_\alpha \,\rangle \begin{Bmatrix} \mathbf{p}_1^{(\alpha)} \\ \mathbf{p}_2^{(\alpha)} \end{Bmatrix} \tag{39}$$

Since $\delta \mathbf{t}_0 = \delta \mathbf{x}_2 - \delta \mathbf{x}_1$ and with (36), (37) and (38) the variation $(\delta \mathbf{x}_s^2 - \delta \bar{\mathbf{x}}^1)$ needed in (24) can be expressed in matrix form as $(\delta \mathbf{x}_s^2 - \delta \bar{\mathbf{x}}^1) = \delta \hat{\mathbf{x}}_s^T \, \mathbf{B}_s(\bar{\xi})$ where

$$\delta \hat{\mathbf{x}}_s^T \, \mathbf{B}_s(\bar{\xi}) = \langle\, \delta \mathbf{x}_s \,,\, \delta \mathbf{x}_1 \,,\, \delta \mathbf{x}_2 \,,\, \delta \mathbf{t}_1 \,,\, \delta \mathbf{t}_2 \,\rangle \begin{Bmatrix} \mathbf{I} \\ -N_1(\bar{\xi})\, \mathbf{I} + \sum_\alpha [\, H_\alpha(\bar{\xi})\, B_\alpha \mathbf{T}^T - \mathbf{p}_1^{(\alpha)}\, \mathbf{n}_0^T \,] \\ -N_2(\bar{\xi})\, \mathbf{I} - \sum_\alpha [\, H_\alpha(\bar{\xi})\, B_\alpha \mathbf{T}^T - \mathbf{p}_1^{(\alpha)}\, \mathbf{n}_0^T \,] \\ \mathbf{p}_2^{(1)}\, \mathbf{n}_0^T \\ \mathbf{p}_2^{(2)}\, \mathbf{n}_0^T \end{Bmatrix} \tag{40}$$

Thus, finally the matrix form of the variation of the penalty form of all contact contribution for the active contact constraints in (24) is given by

$$\mathrm{d}\Pi_\varepsilon^h = \sum_{s=1}^{n_c} \delta \hat{\mathbf{x}}_s^T \, [\, \varepsilon \, A_s \, \mathbf{B}_s(\bar{\xi}) \, (\mathbf{x}_s^2 - \bar{\mathbf{x}}^1) \,] \tag{41}$$

The linearization follows with (40) from (27) as

$$\mathrm{Dd}\Pi_\varepsilon^h = \sum_{s=1}^{n_c} \varepsilon \, A_s \left[\, \delta \hat{\mathbf{x}}_s^T \, \mathbf{B}_s(\bar{\xi}) \, \mathbf{A}(\bar{\xi}) \, \mathbf{B}_s^T(\bar{\xi}) \, \Delta \hat{\mathbf{x}}_s - (\mathbf{x}_s^2 - \bar{\mathbf{x}}^1)^T \, \Delta \delta \bar{\mathbf{x}}^1 \right] \tag{42}$$

where the last term now has to be derived as a function of the unknown variables. Not that in (42) the linearization $(\Delta \mathbf{x}_s^2 - \Delta \bar{\mathbf{x}}^1) = \mathbf{B}_s^T(\bar{\xi}) \, \Delta \hat{\mathbf{x}}_s$ has been used according to (15).

The last term in (42) results from the linearization of the variation (36) with regard to the variables. This yields

$$(\mathbf{x}_s^2 - \bar{\mathbf{x}}^1)^T \, \Delta \delta \bar{\mathbf{x}}^1 = (\mathbf{x}_s^2 - \bar{\mathbf{x}}^1)^T \sum_{\alpha=1}^{2} H_\alpha(\xi) \; [\, \Delta \delta B_\alpha \, \mathbf{n}_0 + \Delta B_\alpha \, \delta \mathbf{n}_0 + \delta B_\alpha \, \Delta \mathbf{n}_0] \tag{43}$$

In this expression the term δB_α is already known and ΔB_α has the same structure. Thus the term which has to be investigated in detail is $\Delta \delta B_\alpha$.

$$\Delta \delta B_\alpha = \Delta \left\{ \frac{1}{2\, \mathbf{t}_\alpha^T \mathbf{t}_0} \; \left[\, (\mathbf{t}_\alpha^T \, \mathbf{T} - B_\alpha \, \mathbf{t}_\alpha^T \,) \, \delta \mathbf{t}_0 + (\mathbf{n}_0^T - B_\alpha \, \mathbf{t}_0^T \,) \, \delta \mathbf{t}_\alpha \,\right] \right\} \tag{44}$$

Using the results already obtained in (39) and by defining the matrices

$$\begin{aligned} \mathbf{m}_{11}^{(\alpha)} &= -\frac{H_\alpha}{2 \mathbf{t}_\alpha^T \, \mathbf{t}_0} [\, \mathbf{T} \, \mathbf{t}_\alpha \, \mathbf{t}_\alpha^T + \mathbf{t}_\alpha \, \mathbf{t}_\alpha^T \, \mathbf{T}^T - 2\, B_\alpha \, \mathbf{t}_\alpha \, \mathbf{t}_\alpha^T \,] \\ \mathbf{m}_{12}^{(\alpha)} &= \frac{H_\alpha}{2 \mathbf{t}_\alpha^T \, \mathbf{t}_0} [\, \mathbf{n}_0 \, \mathbf{t}_\alpha^T - \mathbf{t}_0 \, \mathbf{t}_\alpha^T \, \mathbf{T}^T - 2\, B_\alpha \, \mathbf{t}_0 \, \mathbf{t}_\alpha^T \,] \\ \mathbf{m}_{22}^{(\alpha)} &= \frac{H_\alpha}{2 \mathbf{t}_\alpha^T \, \mathbf{t}_0} [\, \mathbf{n}_0 \, \mathbf{t}_0^T + \mathbf{t}_0 \, \mathbf{n}_0^T - 2\, B_\alpha \, \mathbf{t}_0 \, \mathbf{t}_0^T \,] \end{aligned} \tag{45}$$

we arrive at

$$H_\alpha \, \Delta\delta B_\alpha = \langle \, \delta \mathbf{t}_0 \, , \delta \mathbf{t}_\alpha \, \rangle \, \mathbf{M}_\alpha \, \begin{Bmatrix} \Delta \mathbf{t}_0 \\ \Delta \mathbf{t}_\alpha \end{Bmatrix} \tag{46}$$

In this expression the matrix $\mathbf{M}_\alpha$ has the structure

$$\mathbf{M}^{(\alpha)} = \frac{1}{2 \, \mathbf{t}_\alpha^T \, \mathbf{t}_0} \begin{bmatrix} \mathbf{m}_{11}^{(\alpha)} & -H_a \, (B_\alpha \mathbf{I} - \mathbf{T}) - \mathbf{m}_{12}^{(\alpha)^T} \\ -H_\alpha \, (B_\alpha \mathbf{I} - \mathbf{T}^T) - \mathbf{m}_{12}^{(\alpha)} & -\mathbf{m}_{22}^{(\alpha)} \end{bmatrix} \tag{47}$$

Furthermore let us define with (38) the 2×2 matrices which are needed to describe the last two terms in (43)

$$\begin{aligned} \mathbf{P}_1^{(\alpha)} &= \mathbf{T} \, (\mathbf{x}_s^2 - \bar{\mathbf{x}}^1) \, \mathbf{p}_1^{(\alpha)^T} \\ \mathbf{P}_2^{(\alpha)} &= \mathbf{T} \, (\mathbf{x}_s^2 - \bar{\mathbf{x}}^1) \, \mathbf{p}_2^{(\alpha)^T} \end{aligned} \tag{48}$$

Now all terms have been derived and the final matrix form of the linearization can be stated

$$(\mathbf{x}_s^2 - \bar{\mathbf{x}}^1)^T \, \Delta\delta\mathbf{x} = \langle \, \delta \mathbf{t}_0 \, , \delta \mathbf{t}_1 \, , \mathbf{t}_2 \, \rangle \, \mathbf{M} \, \begin{Bmatrix} \Delta \mathbf{t}_0 \\ \Delta \mathbf{t}_1 \\ \Delta \mathbf{t}_2 \end{Bmatrix} \tag{49}$$

Here the matrix $\mathbf{M}$ is given with (47) and (48) as

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{11}^{(1)} + \mathbf{M}_{11}^{(2)} + \mathbf{P}_1^{(1)} + \mathbf{P}_1^{(1)^T} + \mathbf{P}_1^{(2)} + \mathbf{P}_1^{(2)^T} & \mathbf{P}_2^{(1)} + \mathbf{M}_{12}^{(1)} & \mathbf{P}_2^{(2)} + \mathbf{M}_{12}^{(2)} \\ \mathbf{P}_2^{(1)^T} + \mathbf{M}_{21}^{(1)} & \mathbf{M}_{22}^{(1)} & \mathbf{0} \\ \mathbf{P}_2^{(2)^T} + \mathbf{M}_{21}^{(2)} & \mathbf{0} & \mathbf{M}_{22}^{(2)} \end{bmatrix} \tag{50}$$

Finally, we have to express the vectors $\mathbf{t}_0 \, , \mathbf{t}_1$ and $\mathbf{t}_2$ in terms of the nodal values $\mathbf{x}_0 \, , \mathbf{x}_1 \, , \mathbf{x}_2$ and $\mathbf{x}_3$. For this we define the transformation

$$\begin{Bmatrix} \delta \mathbf{t}_0 \\ \delta \mathbf{t}_1 \\ \delta \mathbf{t}_2 \end{Bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \delta \mathbf{x}_s \\ \delta \mathbf{x}_0 \\ \delta \mathbf{x}_1 \\ \delta \mathbf{x}_2 \\ \delta \, \mathbf{x}_3 \end{Bmatrix} \tag{51}$$

where $\delta \mathbf{x}_s$ has been added for completeness. This transformation can now be applied in (49) and yields together with (50) the final matrix form for the linearization of the normal gap in (42).

3.2 Smooth Bezier interpolation

Bezier polynomials which are introduced here to obtain a continuous normal field are cubic functions. As the Hermitian functions these are defined by four points on the master surface, however in a different manner, see Figure 4.

The Bezier interpolation for the segment described by node 1 and 2 yields

$$\mathbf{x}(\xi) = B_1(\xi)\mathbf{x}_1 + B_2(\xi)\mathbf{x}_{1+} + B_3(\xi)\mathbf{x}_{2-} + B_4(\xi)\mathbf{x}_2 \tag{52}$$

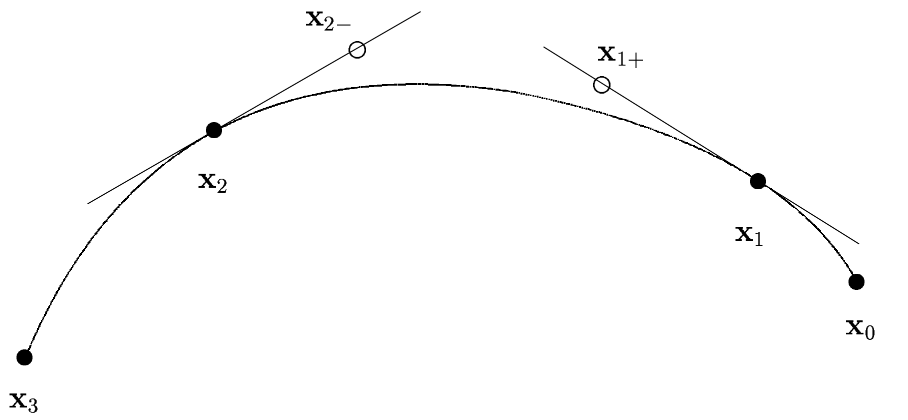

Figure 4: C^1 – continuous Bezier interpolation of contact surface.

where the Bezier interpolation functions are defined as

$$\begin{aligned} B_1(\xi) &= \frac{1}{8}\,(\,1-\xi\,)^3 \\ B_2(\xi) &= \frac{3}{8}\,(\,1-\xi\,)^2\,(\,1+\xi\,) \\ B_3(\xi) &= \frac{3}{8}\,(\,1-\xi\,)\,(\,1+\xi\,)^2 \\ B_4(\xi) &= \frac{1}{8}\,(\,1+\xi\,)^3 \end{aligned} \tag{53}$$

Observe that the interpolation lies in the convex hull spanned by the nodes $\mathbf{x}_1\,,\mathbf{x}_{1+}\,,\mathbf{x}_{2-}$ and $\mathbf{x}_2$.

Our main requirement for the interpolation is, that the tangent vectors of adjacent segments have to be equal to maintain C^1 continuity over segment boundaries. This condition can be applied to compute the interior points of the segment $\mathbf{x}_{1+}\,,\mathbf{x}_{2-}$. By defining the tangent vectors at nodes 1 and 2 as in the previous section we obtain

$$\mathbf{t}_1 = \frac{\alpha}{2}\,(\mathbf{x}_2 - \mathbf{x}_0) \quad \text{and} \quad \mathbf{t}_2 = \frac{\alpha}{2}\,(\mathbf{x}_3 - \mathbf{x}_1) \tag{54}$$

Now we take the derivative of (52) and evaluate this at the end points $\xi = -1$ and $\xi = +1$. By setting this equal to the tangent vectors $\mathbf{t}_\alpha$ we obtain

$$\begin{aligned} \mathbf{x}_{1+} &= \mathbf{x}_1 - \frac{\alpha}{2}\,(\mathbf{x}_2 - \mathbf{x}_0) \\ \mathbf{x}_{2-} &= \mathbf{x}_2 + \frac{\alpha}{2}\,(\mathbf{x}_3 - \mathbf{x}_1) \end{aligned} \tag{55}$$

Here α is a parameter which specifies how far nodes $\mathbf{x}_{1+}$ and $\mathbf{x}_{2-}$ are away from nodes $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively. For different α the shape of the surface interpolation changes. In the limit for $\alpha \to 0$ we obtain an almost flat segment, however the corner region between adjacent segments is still C^1 continuous. Since the shape of the surface changes during the finite deformation process, α might be adapted within the calculation. However a good choice for α is $\alpha = \frac{1}{3}$, see also [5].

With (55) we can rewrite the interpolation (52). This leads to

$$\mathbf{x}(\xi) = \sum_{i=0}^{3} \bar{B}_i(\xi)\,\mathbf{x}_i \tag{56}$$

with

$$\begin{aligned} \bar{B}_0(\xi) &= \frac{\alpha}{2}\,B_2(\xi) \\ \bar{B}_1(\xi) &= B_1(\xi) + B_2(\xi) - \frac{\alpha}{2}\,B_3(\xi) \\ \bar{B}_2(\xi) &= B_3(\xi) + B_4(\xi) - \frac{\alpha}{2}\,B_2(\xi) \\ \bar{B}_3(\xi) &= \frac{\alpha}{2}\,B_3(\xi) \end{aligned} \tag{57}$$

Now we compute the first and second derivative of $\mathbf{x}(\xi)$ with respect to the surface coordinate ξ for later use

$$\begin{aligned} \mathbf{x}_{,\xi}(\xi) &= \sum_{i=0}^{3} \bar{B}_{i\,,\xi}(\xi)\,\mathbf{x}_i \\ \mathbf{x}_{,\xi\xi}(\xi) &= \sum_{i=0}^{3} \bar{B}_{i\,,\xi\xi}(\xi)\,\mathbf{x}_i \end{aligned} \tag{58}$$

The expression for the variation of the gap (13) using this interpolation is now

$$\mathrm{d}g_{N_s} = \left[\delta\mathbf{x}_s - \sum_{i=0}^{3} \bar{B}_i(\bar{\xi})\,\delta\mathbf{x}_i\right]\cdot\bar{\mathbf{n}}^1 \tag{59}$$

which easily is expressed in matrix form as

$$\mathrm{d}g_{N_s} = \delta\hat{\mathbf{x}}_s^T\,\mathbf{B}_n(\bar{\xi}) = \langle\, \delta\mathbf{x}_s^T\,, \delta\mathbf{x}_0^T\,, \delta\mathbf{x}_1^T\,, \delta\mathbf{x}_2^T\,, \delta\mathbf{x}_3^T\,\rangle \begin{Bmatrix} \bar{\mathbf{n}}^1 \\ -\bar{B}_0(\bar{\xi})\,\bar{\mathbf{n}}^1 \\ -\bar{B}_1(\bar{\xi})\,\bar{\mathbf{n}}^1 \\ -\bar{B}_2(\bar{\xi})\,\bar{\mathbf{n}}^1 \\ -\bar{B}_3(\bar{\xi})\,\bar{\mathbf{n}}^1 \end{Bmatrix} \tag{60}$$

Thus the residuum connected with the smooth Bezier contact formulation yields

$$\mathrm{d}\,\Pi_\varepsilon^h = \sum_{s=1}^{n_c} \delta\hat{\mathbf{x}}_s^T\,[\varepsilon\,A_s\,g_{N_s}\,\mathbf{B}_n(\bar{\xi})\,] \tag{61}$$

The linearization of the variation of the gap function needed in (14) can be derived from (18). For this we have to express $\delta\xi$ and $\Delta\xi$, see (21), in matrix form as well as $\delta\bar{\mathbf{x}}^1_{,\xi}$ and $\Delta\bar{\mathbf{x}}^1_{,\xi}$.

Let us first compute the variation of $\bar{\mathbf{x}}^1_{,\xi}$ which yields

$$\delta\bar{\mathbf{x}}^1_{,\xi} = \delta\hat{\mathbf{x}}_s^T\,\mathbf{B}_{n\,,\xi}(\bar{\xi}) = \langle\,\delta\mathbf{x}_s^T\,,\delta\mathbf{x}_0^T\,,\delta\mathbf{x}_1^T\,,\delta\mathbf{x}_2^T\,,\delta\mathbf{x}_3^T\,\rangle \begin{Bmatrix} \mathbf{0} \\ \bar{B}_{0\,,\xi}(\bar{\xi})\,\bar{\mathbf{n}}^1 \\ \bar{B}_{1\,,\xi}(\bar{\xi})\,\bar{\mathbf{n}}^1 \\ \bar{B}_{2\,,\xi}(\bar{\xi})\,\bar{\mathbf{n}}^1 \\ \bar{B}_{3\,,\xi}(\bar{\xi})\,\bar{\mathbf{n}}^1 \end{Bmatrix} \tag{62}$$

Now we can express the first term in (18). Using (19) we obtain

$$\begin{aligned} g_{N_s}\,\mathrm{d}\bar{\mathbf{n}}^1\cdot\mathrm{D}\bar{\mathbf{n}}^1 &= \frac{g_{N_s}}{\|\,\bar{\mathbf{x}}^1_{,\xi}\,\|^2}\,\delta\,\bar{\mathbf{x}}^1_{,\xi}\cdot[\,\bar{\mathbf{n}}^1\otimes\bar{\mathbf{n}}^1\,]\,\Delta\,\bar{\mathbf{x}}^1_{,\xi} \\ &= \delta\hat{\mathbf{x}}_s^T\left[\frac{g_{N_s}}{\|\,\bar{\mathbf{x}}^1_{,\xi}\,\|^2}\,\mathbf{B}_{n\,,\xi}(\bar{\xi})\,\mathbf{B}_{n\,,\xi}(\bar{\xi})^T\right]\Delta\hat{\mathbf{x}}_s \end{aligned} \tag{63}$$

Furthermore we define the matrix form of $(\delta\mathbf{x}_s^2-\delta\bar{\mathbf{x}}^1)\cdot\bar{\mathbf{x}}^1_{,\xi}$ which is needed to compute $\delta\xi$

$$(\delta\mathbf{x}_s^2-\delta\bar{\mathbf{x}}^1)\cdot\bar{\mathbf{x}}^1_{,\xi} = \delta\hat{\mathbf{x}}_s^T\,\mathbf{B}_t(\bar{\xi}) = \langle\,\delta\mathbf{x}_s^T\,,\delta\mathbf{x}_0^T\,,\delta\mathbf{x}_1^T\,,\delta\mathbf{x}_2^T\,,\delta\mathbf{x}_3^T\,\rangle \begin{Bmatrix} \bar{\mathbf{x}}^1_{,\xi} \\ -\bar{B}_0(\bar{\xi})\,\bar{\mathbf{x}}^1_{,\xi} \\ -\bar{B}_1(\bar{\xi})\,\bar{\mathbf{x}}^1_{,\xi} \\ -\bar{B}_2(\bar{\xi})\,\bar{\mathbf{x}}^1_{,\xi} \\ -\bar{B}_3(\bar{\xi})\,\bar{\mathbf{x}}^1_{,\xi} \end{Bmatrix} \tag{64}$$

The variation of the surface coordinate follows now with (64) and (62) from (21) in matrix notation

$$\delta\xi = \delta\hat{\mathbf{x}}_s^T\,\left[\,H_{\xi\xi}\,(\,\mathbf{B}_t(\bar{\xi})+g_{N_s}\,\mathbf{B}_{n\,,\xi}\,)\,\right] = \delta\hat{\mathbf{x}}_s^T\,\mathbf{B}_\xi(\bar{\xi}) \tag{65}$$

where $H_{\xi\xi}=1\,/\,(\,\bar{\mathbf{x}}^1_{,\xi}\cdot\bar{\mathbf{x}}^1_{,\xi}-g_{N_s}\,\bar{\mathbf{n}}^1\cdot\bar{\mathbf{x}}^1_{,\xi\xi}\,)$.

The matrix form of the linearization of the gap function (18) can be expressed with (63) and (65). Thus we obtain finally for (14)

$$\begin{aligned} \mathrm{Dd}\Pi_\varepsilon^h &= \sum_{s=1}^{n_c}\varepsilon\,A_s\,\delta\hat{\mathbf{x}}_s^T\,[\,\mathbf{B}_n(\bar{\xi})\,\mathbf{B}_n(\bar{\xi})^T \\ &- g_{N_s}\,(\,\mathbf{B}_{n\,,\xi}(\bar{\xi})\,\mathbf{B}_\xi(\bar{\xi})^T+\mathbf{B}_\xi(\bar{\xi})\,\mathbf{B}_{n\,,\xi}(\bar{\xi})^T+(\bar{\mathbf{x}}^1_{,\xi\xi}\cdot\bar{\mathbf{n}}^1)\,\mathbf{B}_\xi(\bar{\xi})\,\mathbf{B}_\xi(\bar{\xi})^T \\ &\qquad -\frac{g_{N_s}}{\|\,\bar{\mathbf{x}}^1_{,\xi}\,\|^2}\,\mathbf{B}_{n\,,\xi}(\bar{\xi})\,\mathbf{B}_{n\,,\xi}(\bar{\xi})^T\,)\,]\,\Delta\hat{\mathbf{x}}_s \end{aligned} \tag{66}$$

which denotes the tangent matrix of the smooth Bezier contact formulation. Note that the last term in (18) disappears since the interpolation is linear in the variables $\bar{\mathbf{x}}^1$.

4 Numerical examples

The above derived interpolations have been implemented in the finite element analysis program FEAP, see [9]. To show the performance of the two different smooth contact discretizations, we make a comparison with the classical node–to–segment contact formulations with straight segments. For the derivation of residuum and tangent matrix for the frictionless case, see e. g. [10].

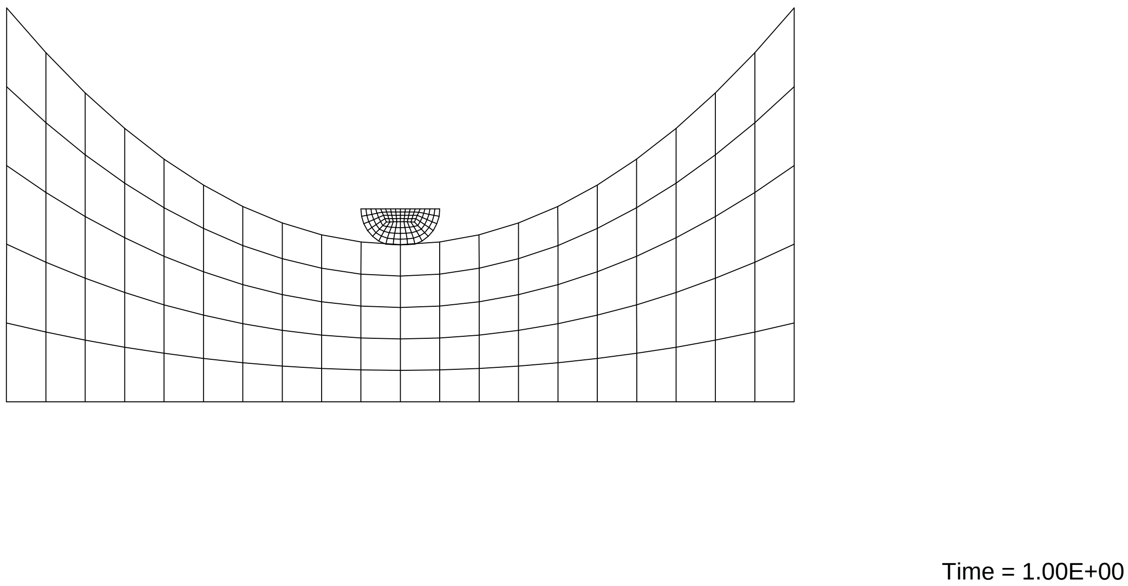

Figure 5: Finite element mesh.

4.1 Sliding of a hemisphere along a parabolic surface

Geometry and mesh of the first example is shown in Fig. 4.1. Here a half of a hemisphere (Radius $r = 1$, bulk modulus $K = 100$ and shear modulus $\mu = 100$) is first pressed onto the parabolic surface (bulk modulus $K = 1000$ and shear modulus $\mu = 1000$) and then slid up the slop.

The final configuration is reached after 36 load steps. It can be seen in Fig. 4.1 which also depicts the principal stresses due to contact in the hemisphere.

The comparison of the smooth interpolation (SNTS) and the standard node–to–segment approach (NTS) yields the following results.

- The total number of iterations to complete 36 load steps plus the initial step is for the smooth contact 181 and for the straight segment contact 207 which means 10 % less computational effort for the smooth contact.

- There are two load steps at which the node–to–segment approach does not converge using standard penalty without special corner treatment. This is due to the jump in the normal in between two adjacent segments.

- The curve of the total horizontal reactions depicted in Fig. 4.1 is smoother when using the SNTS interpolation. This obvious result stems from the better interpolation of the surface geometry.

In total, the smooth contact interpolation is more robust and also more sufficient than the interpolation using straight segments.

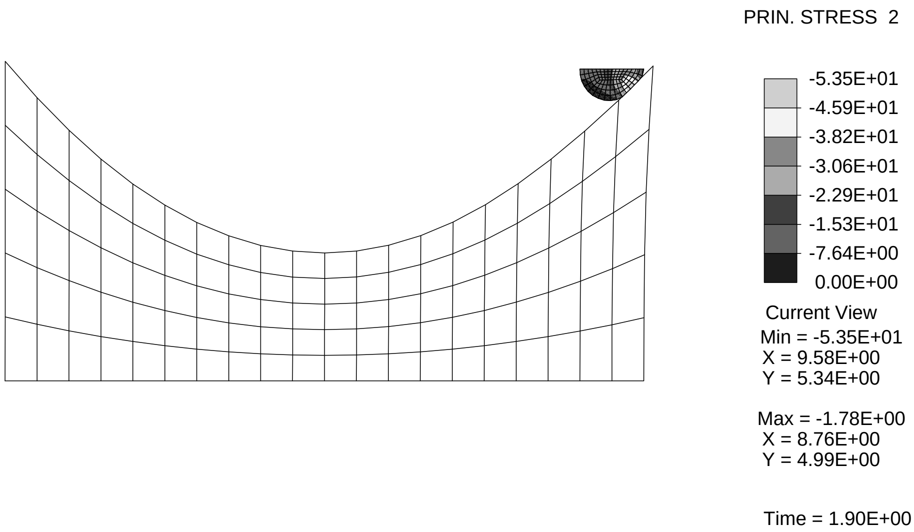

Figure 6: Principal stresses in the hemisphere.

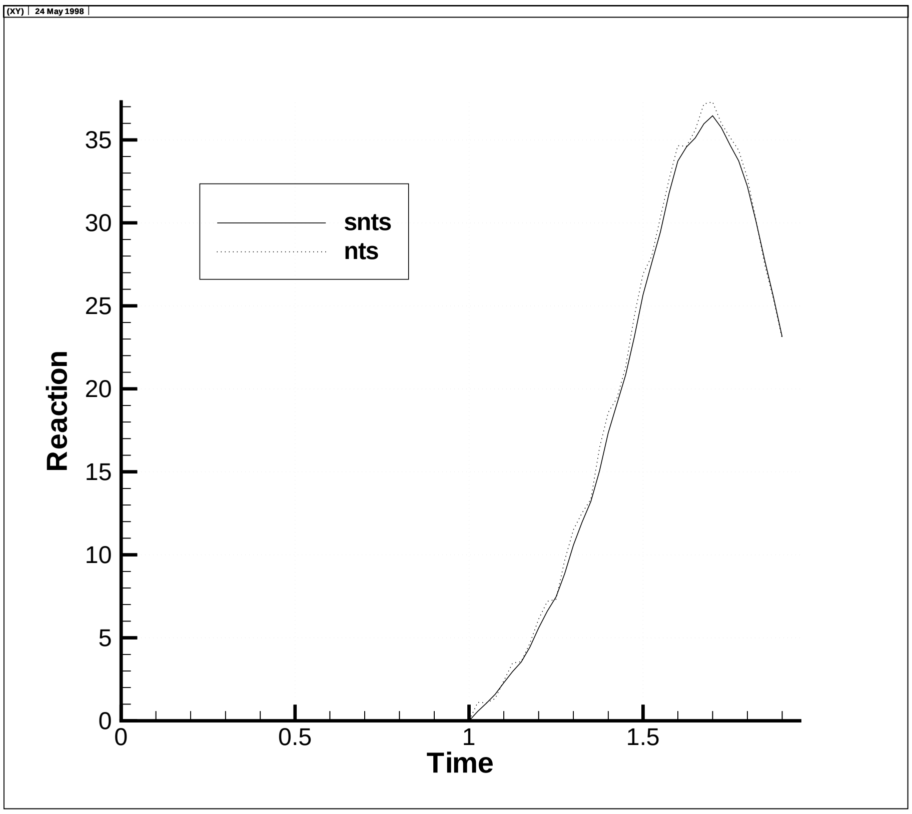

Figure 7: Horizontal reaction force of smooth (SNTS) and non–smooth (NTS) interpolation.

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