Solution of Clamped Rectangular Plate Problems *

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SUMMARY

In this brief note we present an efficient scheme for determining very accurate solutions to the clamped rectangular plate problem. The method is based upon the classical double cosine series expansion and an exploitation of the Sherman-Morrison-Woodbury formula. If the cosine expansion involves M terms and N terms in the two plate axes directions, then the classical method method for this problem involves solving a system of $(MN) \times (MN)$ equations. Our proposal reduces the problem down to a system of well conditioned $N \times N$ equations (or $M \times M$ when $M < N$). Numerical solutions for rectangular plates with various side ratios are presented and compared to the solution generated via Hencky's method. Corrections to classical results and additional digits for use in finite element convergence studies are given. As an application example, these are used to show the rate of convergence for thin plate finite element solutions computed using the Bogner-Fox-Schmit element.

KEY WORDS: Clamped Plate

1. Introduction

When developing new finite elements for solution of plate problems based on the Reissner-Mindlin theory it is necessary to check for locking at the thin plate limit to ensure proper behavior. Generally, the most severe test is for problems where all boundary parameters are restrained. Such a situtation leads to a need for highly accurate solutions with all boundary points set for clamped conditions. For clamped rectangular thin plates no accurate results appear to be available. Here we present a method for obtaining such solutions and tabulate the results for various aspect ratios. In the process, we also correct some commonally referenced but incorrect numerical values.

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An approximate solution for a clamped rectangular thin plate problem subjected to uniform loading has been presented by many authors [1, 2, 3, 4, 5]. This problem requires the solution of the differential equation

$$D\,\nabla^4 w = q \tag{1}$$

subject to the boundary conditions

$$w = 0 \quad \text{and} \quad \frac{\partial w}{\partial n} = 0 \tag{2}$$

along all edges. In (1), ∇^4 denotes the bi-harmonic operator, q is a constant loading normal to the plate and D a constant plate stiffness given by

$$D = \frac{E h^3}{12(1-\nu^2)} \tag{3}$$

with E the modulus of elasticity, ν Poisson's ratio and h the plate thickness.

An approximate solution to the problem may be constructed by finite element methods. However, in order to establish rates of convergence for these it is necessary to know a very accurate solution to the problem. Two methods of solution that have been proposed are the double cosine series (see e.g. [6]‡) and Hencky's method (see e.g. [1, 7, 8, 9, 10]). Hencky's method is well known to converge quickly but does pose some slightly tricky issues with regard to programming due to over/underflow problems in the evaluation of hyperbolic trigonometric functions with large arguments. The double cosine series method, on the other hand, is devoid of the over/underflow issue but is known to converge very slowly. Here we consider a method for efficently determining a very large number of terms in the double cosine series. The series is given by

$$w(x,y) = \sum_{m=1}^{M}\sum_{n=1}^{N}(1-\cos 2m\pi x/a)(1-\cos 2n\pi y/b)\,w_{mn} \quad , \tag{4}$$

where a and b are side lengths in the $0 \le x \le a$ and $0 \le y \le b$ directions, respectively, and w_{mn} are parameters to be determined. Each term of this series satisfies the boundary conditions (2).

The parameters w_{mn} appearing in (4) may be determined using a Ritz method (or alternatively, a Galerkin process) with the functional

$$\Pi(w) = \frac{1}{2}\int_A D\,\left[\nabla^2 w\right]^2 \,\mathrm{d}A - \int_A w\,q\,\mathrm{d}A = \min\ . \tag{5}$$

To carry out the computations we use the transformations

$$\begin{aligned} x &= a\,\xi \quad ; \quad 0 \le \xi \le 1 \\ y &= b\,\eta \quad ; \quad 0 \le \eta \le 1 \end{aligned} \tag{6}$$

and rewrite the functional as $\widehat{\Pi} = \Pi/(ab)$; i.e.

$$\widehat{\Pi}(w) = \frac{1}{2}\frac{D}{a^4}\int_0^1\int_0^1\left[\frac{\partial^2 w}{\partial \xi^2} + (\frac{a}{b})^2\frac{\partial^2 w}{\partial \eta^2}\right]^2 \mathrm{d}A - \int_0^1\int_0^1 w\,q\,\mathrm{d}\xi\,\mathrm{d}\eta = \min\ . \tag{7}$$

‡Szilard mistakenly includes only the odd order terms in each series.

The series now is given in nondimensional form as

$$w(\xi,\eta) = \sum_{m=1}^{M}\sum_{n=1}^{N}(1-\cos 2m\pi\xi)(1-\cos 2n\pi\eta)\, w_{mn} \tag{8}$$

Substitution of (8) into (7) and evaluating the integrals leads to a set of linear algebraic equations which may be written symbolically as

$$\mathbf{K}\,\mathbf{w} = \mathbf{b}\ . \tag{9}$$

Ordering the unknown parameters as

$$\mathbf{w} = [w_{11}, w_{21}, \dots w_{M1}, w_{12}, w_{22}, \dots w_{MN}]^T\ , \tag{10}$$

the non-zero terms in $\mathbf{K}$ are given by

$$\begin{aligned}
K^{(1)}_{ii} &= \left[m^2 + (\tfrac{a}{b})^2 n^2\right]^2 && ;\ \ i = M(n-1)+m \\
K^{(2)}_{ij} &= 2\,m^4 && ;\ \ i = M(n-1)+m \\
& && \quad\ \ j = M(p-1)+m\ ;\ p = 1,2,\dots,N \\
K^{(3)}_{ij} &= 2\,\left(\tfrac{a}{b}\,n\right)^4 && ;\ \ i = M(n-1)+m \\
& && \quad\ \ j = M(n-1)+p\ ;\ p = 1,2,\dots,M
\end{aligned} \tag{11}$$

with the total accumulated as

$$\mathbf{K} = \mathbf{K}^{(1)} + \mathbf{K}^{(2)} + \mathbf{K}^{(3)}\ . \tag{12}$$

Finally, the right hand side is given by

$$\mathbf{b} = \frac{q\,a^4}{4\,\pi^4 D}\,\mathbf{e}^\star\ , \tag{13}$$

where $\mathbf{e}^\star$ is a vector with all unit entries.

2. Solution

The above series solution converges quite slowly and to obtain results for displacements, moments, and energy of sufficient accuracy to assess convergence of finite element solutions it is necessary to use about one million terms. Without some simplifications the solution of (13) can be very time consuming. Here we propose to exploit the structure of the sparse matrix to solve the equations using a symmetric form of the Sherman-Morrison-Woodbury formula [11]. First, we write the coefficient matrix in the form

$$\mathbf{K} = \mathbf{A} + \mathbf{U}\mathbf{D}\mathbf{U}^T \tag{14}$$

in which $\mathbf{A}$ and $\mathbf{D}$ are diagonal matrices and $\mathbf{U}$ is a rectangular matrix with structure

$$\mathbf{U} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{1} & \dots & \mathbf{0} & \mathbf{I} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{1} & \mathbf{I} \end{bmatrix}_{(MN)\times(M+N)} \tag{15}$$

where $\mathbf{1}$ is a unit column vector of length M, $\mathbf{0}$ is a zero column vector of length M and $\mathbf{I}$ is an $M \times M$ square matrix.

With the above structure, the solution for the parameters may be determined from

$$\left(\mathbf{A} + \mathbf{U}\,\mathbf{D}\,\mathbf{U}^T\right)\,\mathbf{w} = \mathbf{b} \quad . \tag{16}$$

Using a symmetric form of the Sherman-Morrison-Woodbury formula the solution may be expressed as

$$\mathbf{w} = \left[\mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{U}\left(\mathbf{D}^{-1} + \mathbf{U}^T\mathbf{A}^{-1}\mathbf{U}\right)^{-1}\mathbf{U}^T\mathbf{A}^{-1}\right]\,\mathbf{b} \quad . \tag{17}$$

The solution is constructed using the steps

1. Set
$$\mathbf{w}_0 = \mathbf{A}^{-1}\mathbf{b} \; .$$
2. Multiply by the transpose of the sparse $\mathbf{U}$ array
$$\mathbf{v}_0 = \mathbf{U}^T\mathbf{w}_0 \; .$$
3. Solve the equations
$$\left(\mathbf{D}^{-1} + \mathbf{U}^T\mathbf{A}^{-1}\mathbf{U}\right)\,\mathbf{v}_1 = \mathbf{v}_0$$
for $\mathbf{v}_1$.
4. Multiply by the sparse array $\mathbf{U}$ to obtain
$$\mathbf{w}_1 = \mathbf{U}\,\mathbf{v}_1 \; .$$
5. Accumulate the solution as
$$\mathbf{w} = \mathbf{w}_0 - \mathbf{A}^{-1}\mathbf{w}_1 \; .$$

The above steps involve inversion of two diagonal arrays ($\mathbf{A}$ and $\mathbf{D}$); multiplication of a vector by $\mathbf{U}$ and its transpose; and the solution to a set of symmetric equations of size $(M+N) \times (M+N)$. The size of this system is in sharp contrast to (9) which involves a system of $(MN) \times (MN)$ equations.

The system of equations to be solved has a very simple structure due to the sparse nature of the $\mathbf{U}$ array. The matrix part may be written as

$$\mathbf{U}^T\mathbf{A}^{-1}\mathbf{U} = \begin{bmatrix} \mathbf{D}_1 & \mathbf{G} \\ \mathbf{G}^T & \mathbf{D}_2 \end{bmatrix} \tag{18}$$

in which $\mathbf{D}_1$ and $\mathbf{D}_2$ are diagonal arrays of length N and M, respectively, and $\mathbf{G}$ is a full matrix of size $N \times M$.

The solution of the set of coupled equations we solve is scaled to have unit diagonals as

$$\mathbf{S}\left(\mathbf{D}^{-1} + \mathbf{U}^T\mathbf{A}^{-1}\mathbf{U}\right)\,\mathbf{S}\,\mathbf{v}_1^{\star} = \mathbf{S}\,\mathbf{v}_0 \; ,$$

where $\mathbf{S}$ is a diagonal matrix defined by

$$\mathbf{S} = \begin{bmatrix} \mathbf{D}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_2 \end{bmatrix}^{-1/2}$$

and the solution vector is recovered from

$$\mathbf{v}_1 = \mathbf{S}\,\mathbf{v}_1^\star \ .$$

With this scaling the resulting coefficient matrix has a very low condition number as can be observed in the next section. Without the scaling the solution of the set of equations without pivoting suffered from round-off errors when coded in double precision (64 bit IEEE floating point convention).

The solution of the above equations may be accomplished even more efficiently using

$$\mathbf{S}\,\mathbf{v}_0 = \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_1^\star = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}$$

which yields the pair of equations

$$\begin{aligned} \mathbf{X}_1 + \mathbf{G}\,\mathbf{X}_2 &= \mathbf{V}_1 \\ \mathbf{X}_2 + \mathbf{G}^T\mathbf{X}_1 &= \mathbf{V}_2 \ . \end{aligned}$$

Substituting the second into the first gives an $N \times N$ set

$$\left[\mathbf{I} - \mathbf{G}\mathbf{G}^T\right]\,\mathbf{X}_1 = \mathbf{V}_1 - \mathbf{G}\,\mathbf{V}_2 \ .$$

After $\mathbf{X}_1$ is computed $\mathbf{X}_2$ is determined from the second equation set.

Once we determine $\mathbf{v}_1$ we can sum the series (11) to obtain the displacement at any point. Similarly, we can obtain the curvatures from

$$\begin{aligned} \frac{\partial^2 w}{\partial x^2} &= \frac{1}{a^2}\,\frac{\partial^2 w}{\partial \xi^2} = \frac{4\,\pi^2}{a^2}\,\sum_{m=1}^{M}\sum_{n=1}^{N} m^2\,\cos 2m\pi\xi\,(1-\cos 2n\pi\eta)\,w_{mn} \ , \\ \frac{\partial^2 w}{\partial y^2} &= \frac{1}{b^2}\,\frac{\partial^2 w}{\partial \eta^2} = \frac{4\,\pi^2}{b^2}\,\sum_{m=1}^{M}\sum_{n=1}^{N} n^2\,(1-\cos 2m\pi\xi)\,\cos 2n\pi\eta\,w_{mn} \ , \\ \frac{\partial^2 w}{\partial x \partial y} &= \frac{1}{ab}\,\frac{\partial^2 w}{\partial \xi \partial \eta} = \frac{4\,\pi^2}{ab}\,\sum_{m=1}^{M}\sum_{n=1}^{N} m\,n\,\sin 2m\pi\xi\,\sin 2n\pi\eta\,w_{mn} \ , \end{aligned} \tag{19}$$

and compute the moments from

$$\begin{aligned} M_x &= -D\,[\frac{\partial^2 w}{\partial x^2} + \nu\,\frac{\partial^2 w}{\partial y^2}] \ , \\ M_y &= -D\,[\frac{\partial^2 w}{\partial y^2} + \nu\,\frac{\partial^2 w}{\partial x^2}] \ , \\ M_{xy} &= -D\,(1-\nu)\frac{\partial^2 w}{\partial x \partial y} \ . \end{aligned} \tag{20}$$

We also compute the work (twice stored energy) by integrating the displacement to obtain

$$E = \int_A q\,w(x,y)\,\mathrm{d}A = a\,b\,\sum_{m=1}^{M}\sum_{n=1}^{N} w_{mn} \ . \tag{21}$$

3. Numerical results

The above solution is applied to a square, clamped plate subjected to uniform loading. In the solution we set $M = N$ and select increasing values for M. The results for the center displacement, center and maximum edge moments and work are tabulated in Table I. We also indicate the condition number of the coefficient matrix for the set of equations used to compute $\mathbf{v}_1$.

Remarks:

1. Shown in the last line of Table I is the result obtained from Hencky's methods using the equations presented by [7]. This solution was computed using a system of 2000×2000 equations. Only the converged digits are shown for this method. While one can see that Hencky's methods gives better accuracy for the same amount of work, we noted that the conditioning of Hencky's method is not at good as the more slowly converging double cosine series equations.
2. We highlight the goodness obtained by scaling the equations (see the condition number) and note that the penultimate row of Table I involves the solution of 4 million unknowns by only solving a 2000×2000 system of equations.
3. Note that a commonly accepted value of $0.0231 q\, a^2$ [5] for the center moment in the square plate is in error in the third digit.
4. Table II presents similar converged results for other aspect ratios. We note that a commonly accepted value of $-0.0571 q\, a^2$ [5, p. 202] for the edge moment (M_y) in the large aspect ratio case is in error in the second digit.
5. In all cases we can see that Hencky's methods gives more significant digits for the same amount of work.

To illustrate the use of the computed values we solve clamped plate problems using the Bogner-Fox-Schmit rectangular finite element [12]. The solution is carried out for the same ratios of the side lengths b/a as in the tables and convergence behavior in the energy norm is shown in Figure 1. To be able to make such convergence studies it is noted that a very accurate "exact" solution is needed. The method presented makes this determination to high accuracy feasible with the double cosine series method.

4. Closure

In this brief note we have presented a method to efficiently compute the solution of the clamped plate problem via the double cosine series method. The technique of exploiting the Sherman-Morrison-Woodbury formula is at the heart of the technique and its successful execution further relies upon a scaling of the governing equations. While, for the particular problem at hand the method of Hencky (essentially a modified Levy method) results in a more satisfactory result, we note that the reduction method developed can be employed for other similar problems using near- (but non-) orthogonal basis expansions.

Table I. Clamped square plate with uniform loading

M	w Center $D/(qa^4)\,10^3$	M_x Edge $10^2/(qa^2)$	M_x Center $10^2/(qa^2)$	E Work $D/(q^2a^4)\,10^4$	Cond. No. κ
200	1.265319036	-5.111075630	2.290436770	3.891200386	3.9208
400	1.265319081	-5.122212116	2.290490957	3.891200726	4.0035
600	1.265319086	-5.125930478	2.290501018	3.891200761	4.0442
800	1.265319087	-5.127790817	2.290504543	3.891200769	4.0702
1000	1.265319087	-5.128907392	2.290506175	3.891200772	4.0890
1200	1.265319087	-5.129651929	2.290507062	3.891200773	—
1400	1.265319087	-5.130183817	2.290507597	3.891200774	—
1600	1.265319087	-5.130582775	2.290507944	3.891200774	—
1800	1.265319087	-5.130893100	2.290508182	3.891200774	—
2000	1.265319087	-5.131141375	2.290508352	3.891200775	—
Hencky	1.265319087	-5.13337648	2.290509078	3.981200775	—

Table II. Converged clamped rectrangular plate solutions with uniform loading for various aspect ratios by solving 2000×2000 equations via the double cosine series method and Hencky's method (both shown to a maximum of 10 converged digits).

b/a	Method	w Center $D/(qa^4)\,10^3$	M_x Edge $10^2/(qa^2)$	M_y Edge $10^2/(qa^2)$	M_x Center $10^2/(qa^2)$	M_y Center $10^2/(qa^2)$	E Work $D/(q^2a^4)\,10^4$
1.2	Double Cosine	1.724870503	-6.38	-5.5	2.99715	2.284043	6.41537043
	Hencky	1.724870503	-6.3897878	-5.5407598	2.9971587	2.2840439	6.41537043
1.4	Double Cosine	2.068143209	-7.25	-5.6	3.49740	2.12663	9.14890620
	Hencky	2.068143209	-7.2591841	-5.6802526	3.4974095	2.1266331	9.14890620
1.6	Double Cosine	2.29996697	-7.80	-5.70	3.81817	1.9250	11.94175880
	Hencky	2.299966977	-7.8033766	-5.709889	3.8181737	1.9250601	11.94175880
1.8	Double Cosine	2.446162656	-8.11	-5.70	4.00944	1.73576	14.73958338
	Hencky	2.446162656	-8.1185893	-5.7066637	4.0094462	1.7357682	14.73958338
2.0	Double Cosine	2.532955769	-8.28	-5.69	4.11549	1.58080	17.53009520
	Hencky	2.532955769	-8.2866062	-5.698664	4.1154990	1.5808029	17.53009520
20.0	Double Cosine	2.60416666	-8.33	-5.6	4.1666	1.25	267.5393
	Hencky	2.604166667	-8.33333	-5.68862	4.166666667	1.250000000	267.5393518

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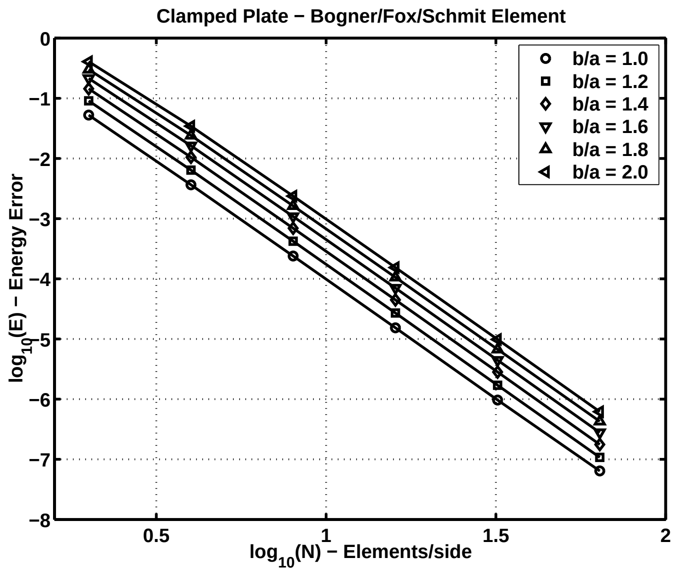

Figure 1. Convergence behavior for clamped rectangular plates

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