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Isogeometric Analysis of Structures Local Treatment of Constraints

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Isogeometric Analysis of Structures

- **This lecture presents:**
  - Summary of current IGA capabilities using *FEAP*.
  - Some methods to treat constraints in IGA.
  - Application to thin structures.
    - Straight and curved beam
    - Comparison with rotation free thin shell (Kiendl, *et al.*, CMAME, vol 198, pp 3902ff, 2009)
  - Graphics for NURBS.

Brief Overview of FEAP

- *FEAP* - Finite Element Analysis Program.
- Research and educational software package developed at University of California, Berkeley.
- Includes element library: Solids, Thermal, Frames, Plates, Membranes & Shells.
- Elements for both small and large deformation analysis.
- Material library for: Elastic, visco-elastic, elasto-plastic, ...
- Solution algorithms by command language statements.
- Screen and hard copy plotting options.
- User module interfaces for elements, meshing, solution, plots.
  - Used for NURBS and T-spline isogeometric solutions.

Isogeometric Modeling

- Isogeometric models described by:
  - Knot vectors (open)
  - Control points and weights
  - Tensor product NURBS (Non-Uniform Rational B-Splines) or T-splines

- IGA elements in *FEAP*
  - Displacement formulations (all *FEAP* solid elements work)
  - Mixed $\mathbf{u} - p - \bar{J}$ formulation (2-d finite deformation only)
  - Thin rotation-free thin shell (Kiendl *et al.*)
  - Some ‘user module’ elements.

- Boundary conditions (only restricted types, no contact)

- Most solution options work (transient, eigenpairs, etc.)

Isogeometric Modeling

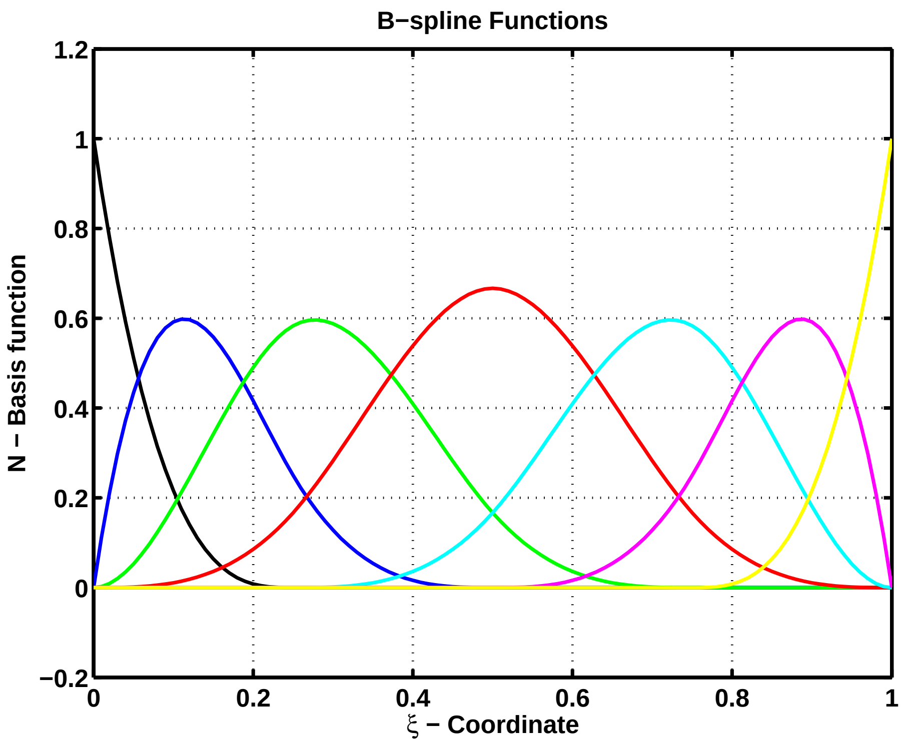
B-Spline interpolations

- Open knot vector: Length m values.

$$\Xi = \Big[\underbrace{\xi_1, \ldots, \xi_1}_{p+1 \text{ times}} \xi_2, \ldots, \underbrace{\xi_r \ldots, \xi_r}_{p+1 \text{ times}}\Big]$$

- Start basis:

$$B_{i,0} = \begin{cases} 1; & \text{if } \xi_i \le \xi < \xi_{i+1} \\ 0; & \text{otherwise} \end{cases}$$



- Recursion n functions:

$$B_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} B_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} B_{i+1,p-1}(\xi)$$

where $n = m - p - 1$ for open knots.
e.g., $\Xi = \left[0,\ 0,\ 0,\ 0,\ \frac{1}{4},\ \frac{1}{2},\ \frac{3}{4}, 1,\ 1,\ 1,\ 1,\right]$, $p = 3$ for figure.

Isogeometric Modeling

- Non-Uniform Rational B-spline (NURBS) defined by

$$N_{i,p}(\xi) = \frac{B_{i,p}(\xi)\, w_i}{\sum_{j=1}^{n} B_{j,p}(\xi)\, w_j}$$

  where w_i are set of n weights defining shape of NURBS.

- Appropriate values of w_i permit description different types of curves; e.g., conic surfaces in addition to polynomials
  - Polynomials in ξ: Weights $w_i = 1$ yields: $N_{i,p}(\xi) = B_{i,p}(\xi)$
  - Circular arc : $\mathbf{x} = N_{1,2}(\xi)\, \tilde{\mathbf{x}}_1 + N_{2,2}(\xi)\, \tilde{\mathbf{x}}_2 + N_{3,2}(\xi)\, \tilde{\mathbf{x}}_3$

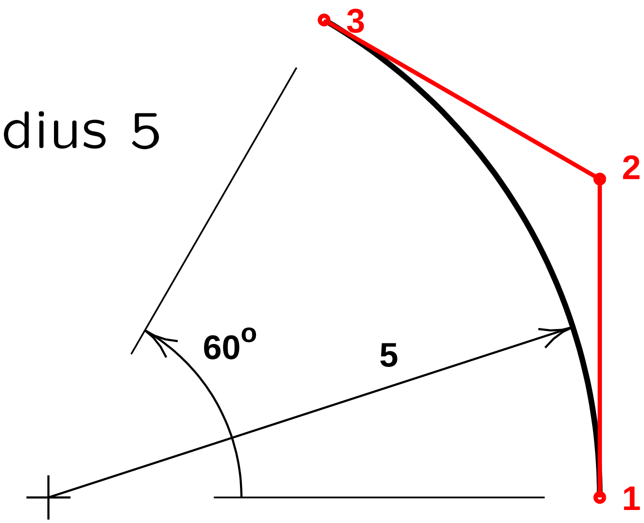
- Example: $\theta = 60^o$ circular arc with radius 5

  $\mathbf{x}_1 = (5,\ 0)$

  $\mathbf{x}_2 = (5,\ \tan(\theta/2))$

  $\mathbf{x}_3 = (5,\ \cos(\theta),\ 5\ \sin(\theta))$

  $(w_1,\ w_2,\ w_3) = (1,\ \cos(\theta/2),\ 1)$



Isogeometric Modeling

Analysis procedure in FEAP

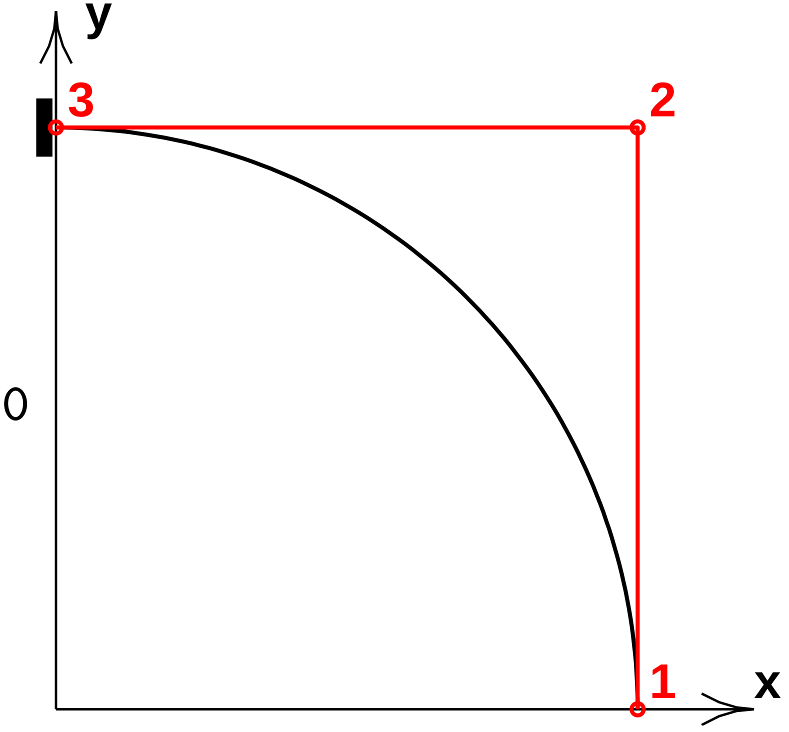
- Define coarse set of control points, knots, 1-d knot-point list, side-patch description:
- Example: Curved beam – input NURBS, knot, side & block

```
NURBs
  1 0 10.0  0.0 1.0
  2 0 10.0 10.0 1.0/sqrt(2)
  3 0  0.0 10.0 1.0

KNOT
  knot 1 0.0 0.0 0.0 1.0 1.0 1.0

NSIDe
  side 1 0 1 1 2 3

NBLOck
  block 1 1 1
```



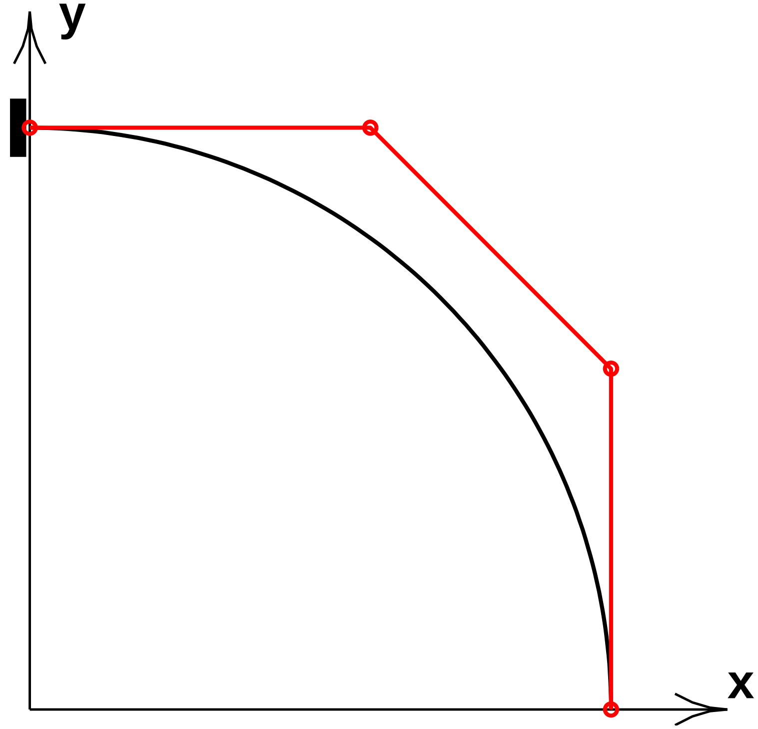
Isogeometric Modeling

- Need to add material properties, loading and boundary conditions. Use standard *FEAP* commands for most.

- Analysis requires degree elevation and knot insertion.

- Example: Elevate order to cubic intepolation

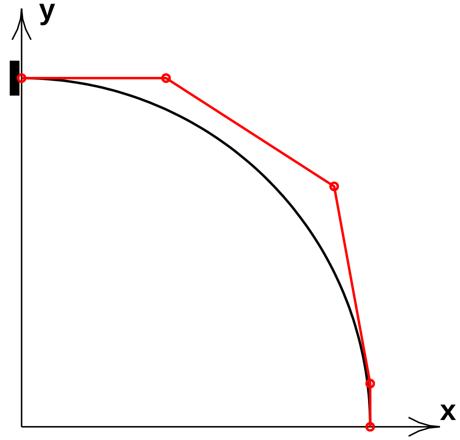
```
BATCh
  ELEVate INITialize
  ELEVate KNOT  1  1
  ELEVate END
END
```



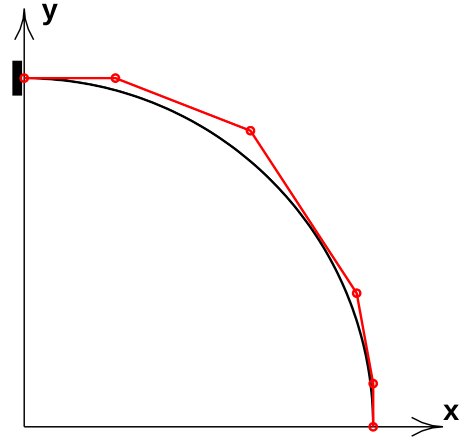
- Knot vector now:
$\Xi = (0,0,0,0,1,1,1,1)$

Isogeometric Modeling

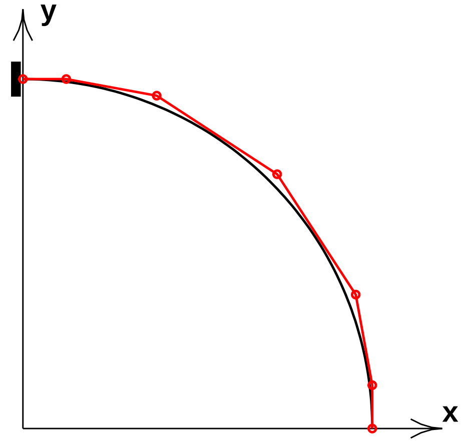
Add knots: k-refinement in circumferential direction



(a) Insert knot 1



(b) Insert knot 2



(c) Insert knot 3: $\Xi = (0, 0, 0, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, 1, 1, 1)$

Isogeometric Modeling

Add knots: k-refinement in FEAP

- Knot insertion performed as

```
BATCh
  INSErt INITialize
  INSErt KNOT knum uu rr
  INSErt END
END
```

  where **uu** knot value & **rr** number times to repeat.

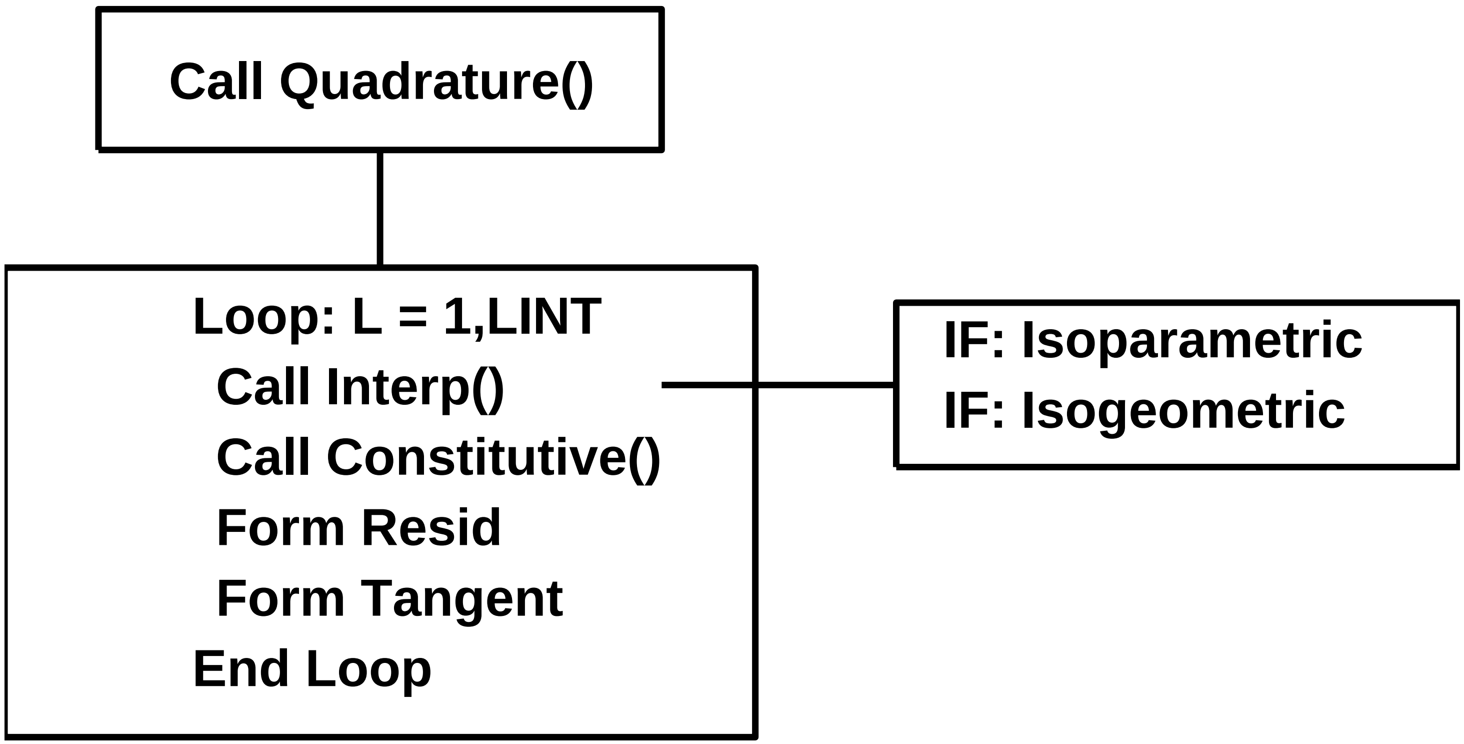
- Each knot insertion lowers continuity by one order.

- Degree elevation and knot insertion create mesh for analysis.

- Elements defined on knot intervals.

Isogeometric Modeling

- Typical displacement formulation element module



- IGA modifies **quadrature** and **interpolation** modules.

Isogeometric Analaysis of Structures

- Displacement methods for IGA standard.

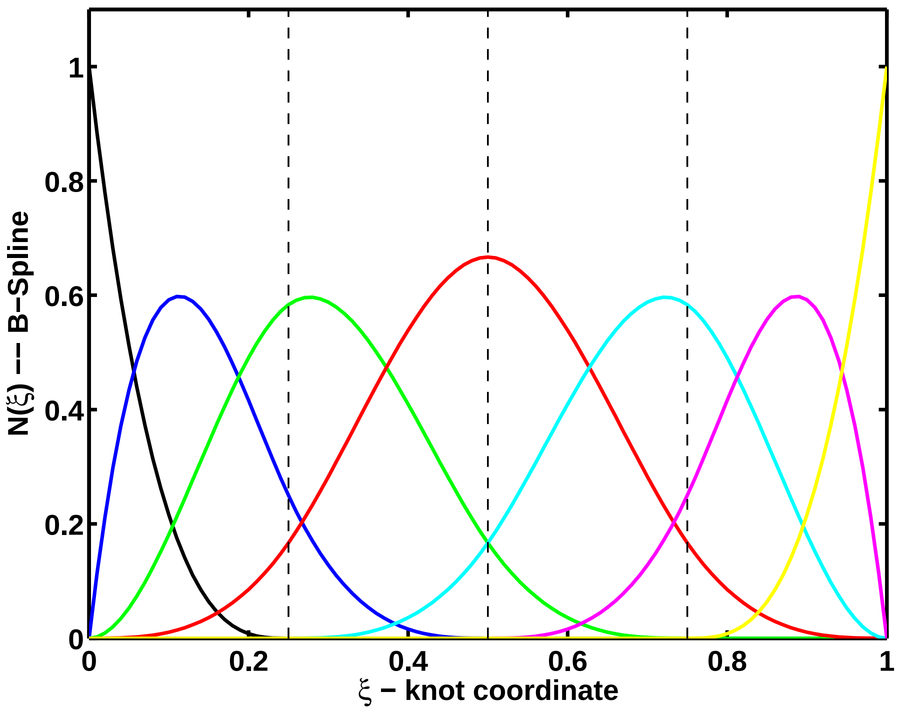
- Solution of structures often involves constraints (shear-bending, membrane-bending, etc.)

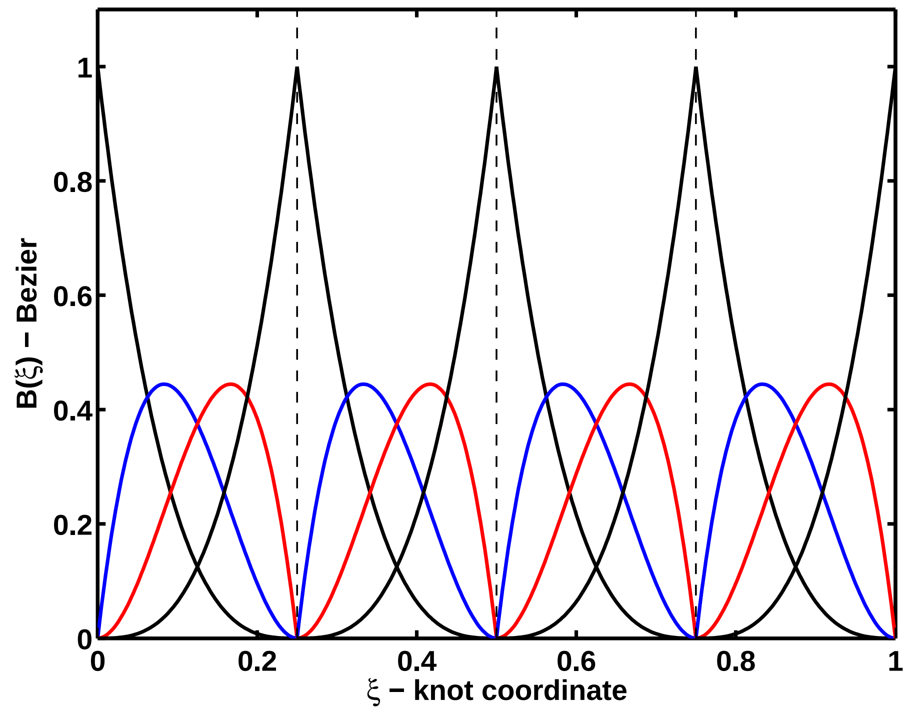
- Known methods to treat constraints
  - Reduced integration
  - Mixed variational methods
    * Hellinger-Reissner: $\mathbf{u} - \boldsymbol{\sigma}$
    * Veubeke-Hu-Washizu: $\mathbf{u} - \boldsymbol{\sigma} - \boldsymbol{\epsilon}$
  - Stabilized methods: GLS, etc.
  - Discrete strain gap: (DSG)

- Bézier extraction provides option for analysis.

Bézier Extraction Form for Elements

- Extraction converts B-splines & NURBS to Bézier form.
  - Example: Cubic B-Spline (elements between dotted lines)





After repeated knot insertion obtain Bézier basis (right figure):

$$\mathbf{N}^e(\xi) = \mathbf{C}^e\mathbf{B}(\xi) \quad \text{where } \mathbf{B}(\xi) \text{ are Bernstein polynomials}$$

- Shape functions: extraction operator times Bernstein polynomials.

Bézier Extraction Form for Elements

- For curves: Interpolations in rational form

$$R_a = \frac{w_a \, B_a(\boldsymbol{\xi})}{W(\boldsymbol{\xi})} \quad \text{where} \quad W(\boldsymbol{\xi}) = \sum_b w_b \, B_b(\boldsymbol{\xi})$$

  where w_a is a weight for the a basis function.

- Permits representation of conics and other curves:
  $w_a = 1$ gives polynomial.

- Surfaces and solids use tensor products of R_a functions.

- NURBS extraction operator form becomes:

$$\boxed{\mathbf{N}^e(\boldsymbol{\xi}) = \mathbf{C}^e \, \mathbf{R}^e(\boldsymbol{\xi})}$$

- Shape function routine given as option to standard FE form.

- Ref: M.J. Borden *et al.*: IJNME, vol. 87, pp 15–47, 2011.

Bézier Extraction Form for Elements

- Coordinate interpolation with extraction operator:

$$\mathbf{x}(\boldsymbol{\xi}) = \underbrace{\sum_{b=1}^{m} \hat{\mathbf{x}}_b \, R_b(\boldsymbol{\xi})}_{\text{Bézier}} = \sum_{b=1}^{m} \underbrace{\sum_{a=1}^{n} \tilde{\mathbf{x}}_a \mathbf{C}_{ab}}_{\hat{\mathbf{x}}_b} \, R_b(\boldsymbol{\xi}) = \sum_{a=1}^{n} \tilde{\mathbf{x}}_a \underbrace{\sum_{b=1}^{m} \mathbf{C}_{ab} \, R_b(\boldsymbol{\xi})}_{N_a(\boldsymbol{\xi})}$$

$$= \underbrace{\sum_{a=1}^{n} \tilde{\mathbf{x}}_a \, N_a(\boldsymbol{\xi})}_{\text{NURBS}} \qquad \text{N.B. } n < m$$

where m number of Bézier points and n number of control points.

- Isoparametric form for other variables:

$$\mathbf{u}(\boldsymbol{\xi}) = \sum_{b=1}^{m} \hat{\mathbf{u}}_b \, R_b(\boldsymbol{\xi}) = \sum_{a=1}^{n} \sum_{b=1}^{m} \tilde{\mathbf{u}}_a \mathbf{C}_{ab} \, R_b(\boldsymbol{\xi}) = \sum_{a=1}^{n} \tilde{\mathbf{u}}_a \, N_a(\boldsymbol{\xi})$$

- Permits element development with either NURBS or Bézier form.

Isogometric Analysis of Structures

- **Some implemented IGA user elements:**

`ELMT`	Description
1	NURBS Euler-Bernoulli beam
2	NURBS 1-d rod.
5	NURBS & T-spline thin C^1 plate
6	NURBS & T-spline thin membrane
7	NURBS & T-spline derivative boundary condition.
8	Bending patch for Euler-Bernoulli beam ties.
18	Global least squares boundary fit for NURBS region
19	Local least squares boundary fit for NURBS region
20	Follower couple to load thin shell element
24	Thin isogeometric non-linear shell (Kiendl et al.)
25	Thin isogeometric linear shell (Kiendl et al.)
43	Linear curved beam with shear deformation.

FEAP user elements for NURBS/T-spline solutions.

Isogometric Analysis of Structures

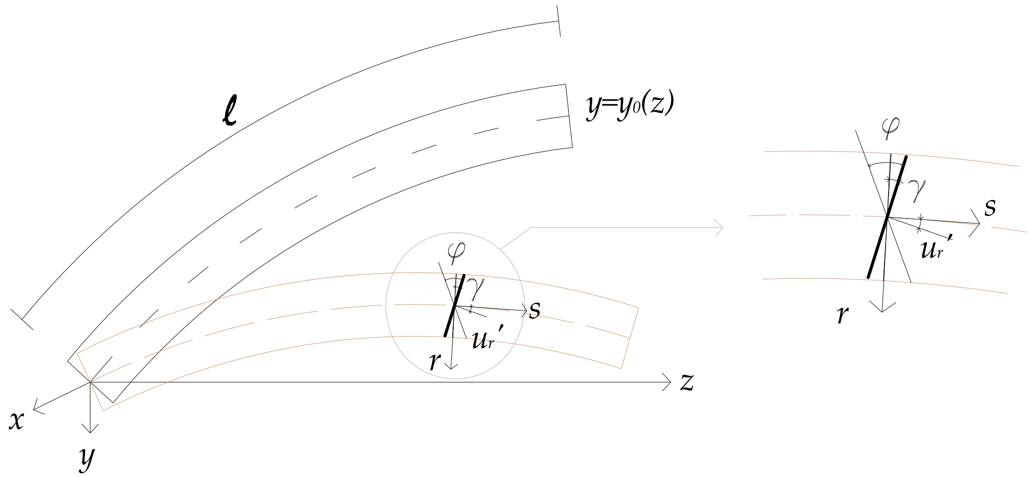
- **Some implemented IGA methods:**

UMACR	Description
6	Extraction operator for 1-d forms. Command: `EXTRact BEZIer n_side`
8	Elevate directions of NURBS block. Command: `ELEVate knot num inc_order`
9	Insert knot in directions of NURBS block. Command: `INSERT knot num u_val times`
2	ZZ projection of stress for 2-d T-splines Command: `NZZP,,n_mat`
0	Elements on edges of NURBS blocks or T-splines. Command: `EIGA,,dir_i x_i`

FEAP commands for NURBS/T-spline solutions.

Isogometric Analysis of Structures

- **Treatment of constraints in structural forms**
  - Example: Shear-bending and membrane-bending in curved rods
- Consider curved 2D Timoshenko beam:



- Strain-displacement relations

$$\epsilon = \frac{\partial u_s}{\partial s} - \frac{u_r}{R} \quad ; \quad \gamma = \frac{u_s}{R} + \frac{\partial u_r}{\partial s} + \varphi \quad ; \quad \kappa = \frac{\partial \varphi}{\partial s}$$

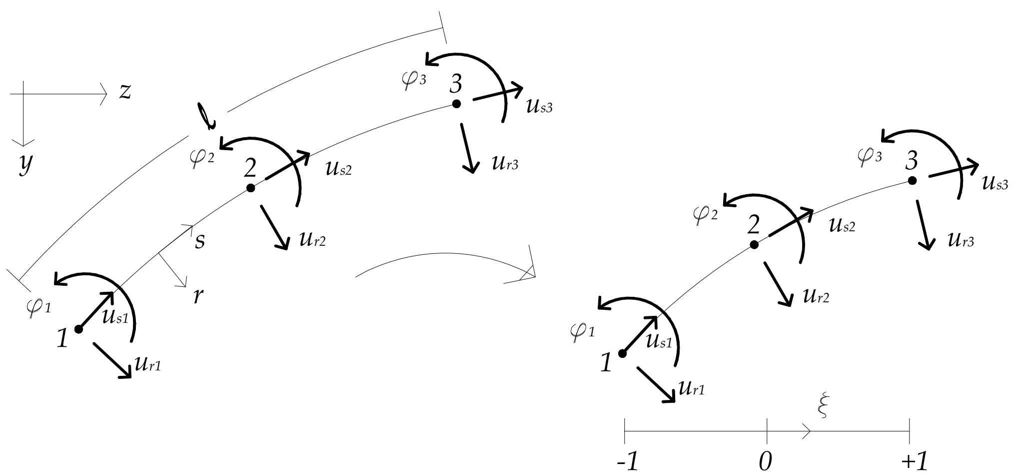
Isogometric Analysis of Structures

- Material constitution (linear elastic)

$$N = EA\epsilon \quad ; \quad V = GA_s\gamma \quad ; \quad M = EJ\kappa$$

- Weak form

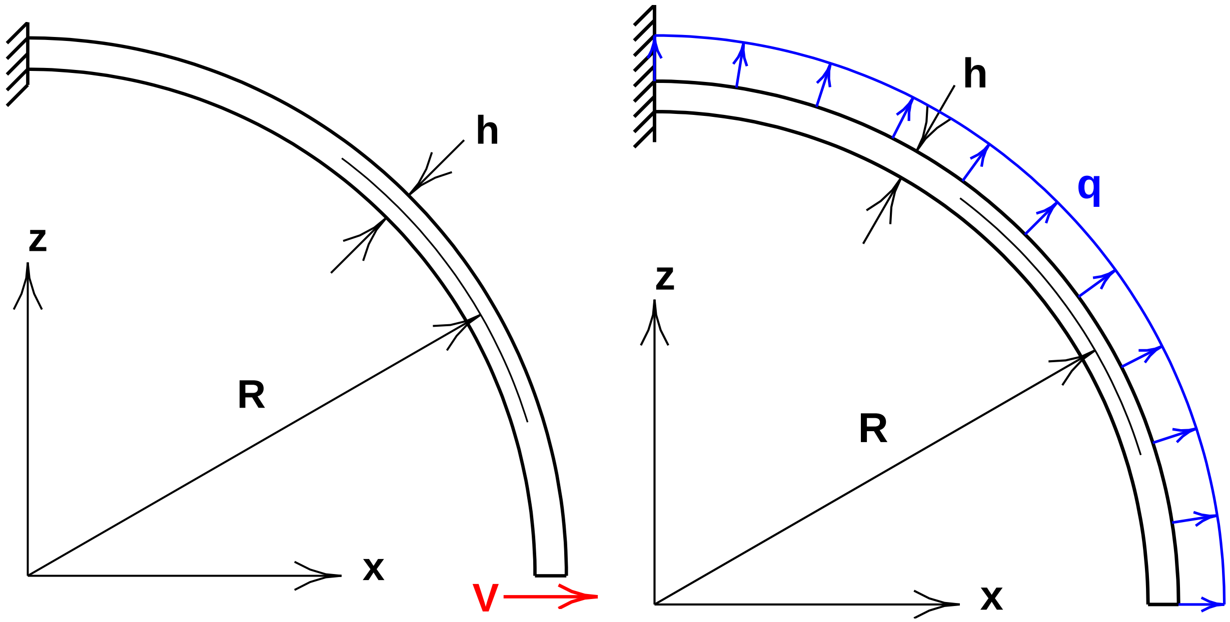
$$\Pi = \int_{-1}^{+1} \left[N\epsilon + M\kappa + V\gamma \right] j(\xi)\, \mathrm{d}\xi + \Pi_{ext}$$



- Unknowns transformed to global coordinates.

Isogometric Analysis of Structures

- **Curved beam problem:**

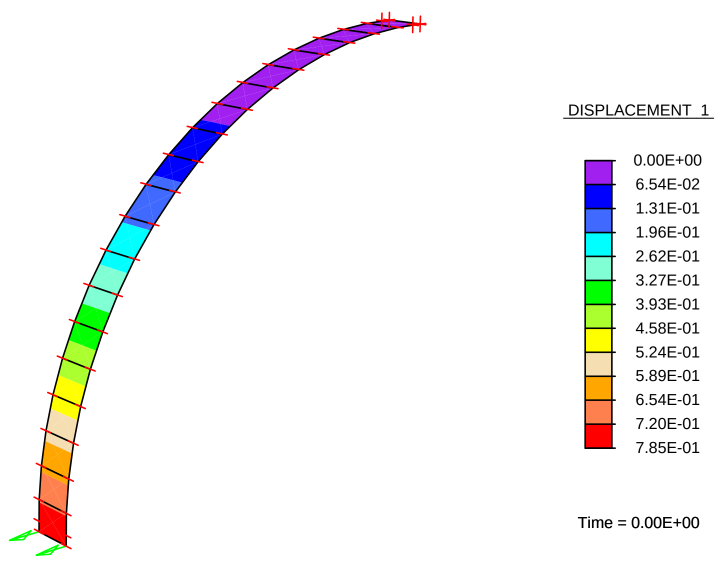


(a) End shear (b) Uniform load

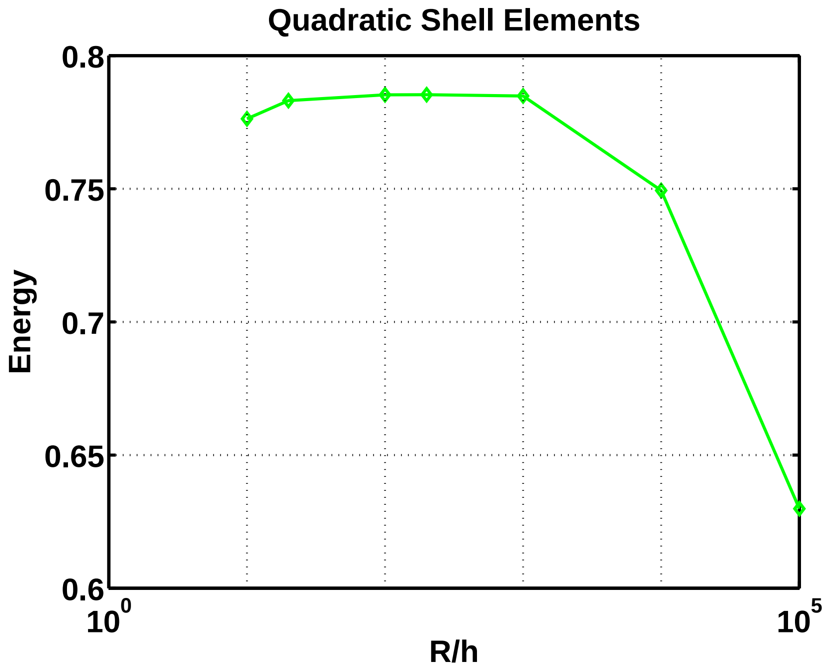
Data: $R = 10$; $h = 0.0001$ to 1; $EI = 1000$; $V = 1$; $q = 0.1$.

Isogometric Analysis of Structures

- **Verification by Kiendl et al. shell:**
  Results for 20-cubic elements with end shear.



(a) u_x contours

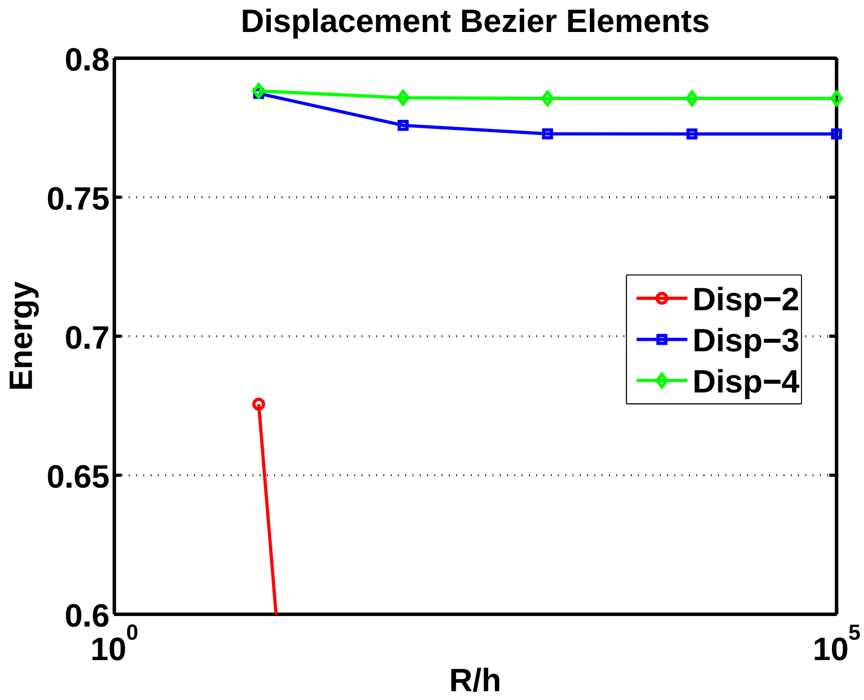


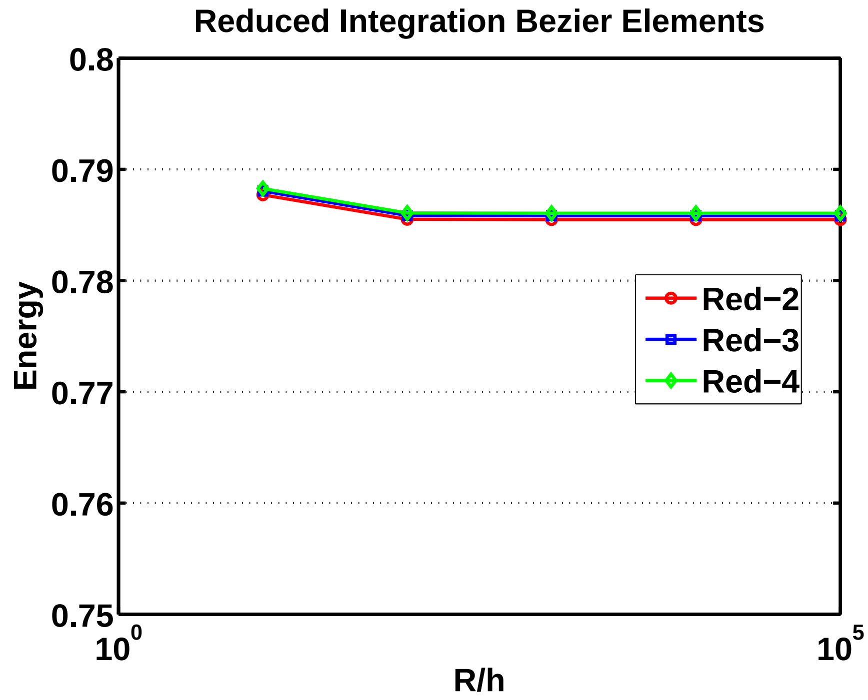
(b) Energy behavior

- Roundoff high for $R/h > 1000$

Isogometric Analysis of Structures

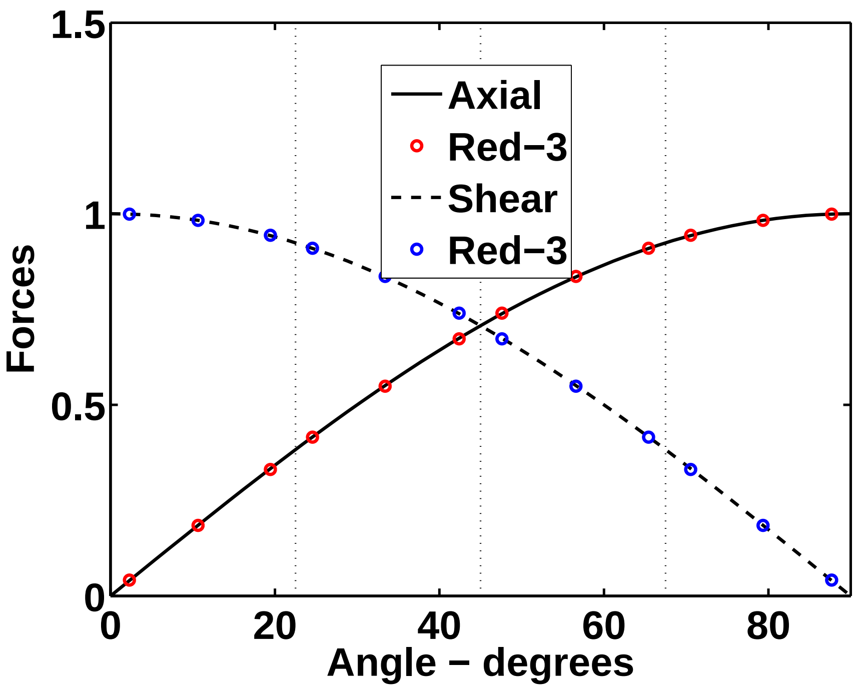
- End shear reduced integration results: 4-elements

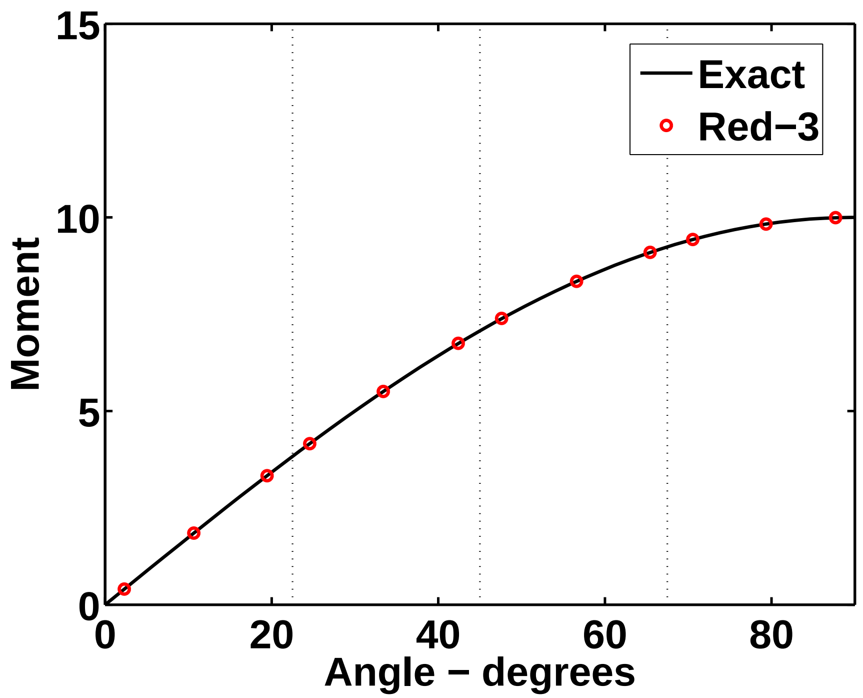




Isogometric Analysis of Structures

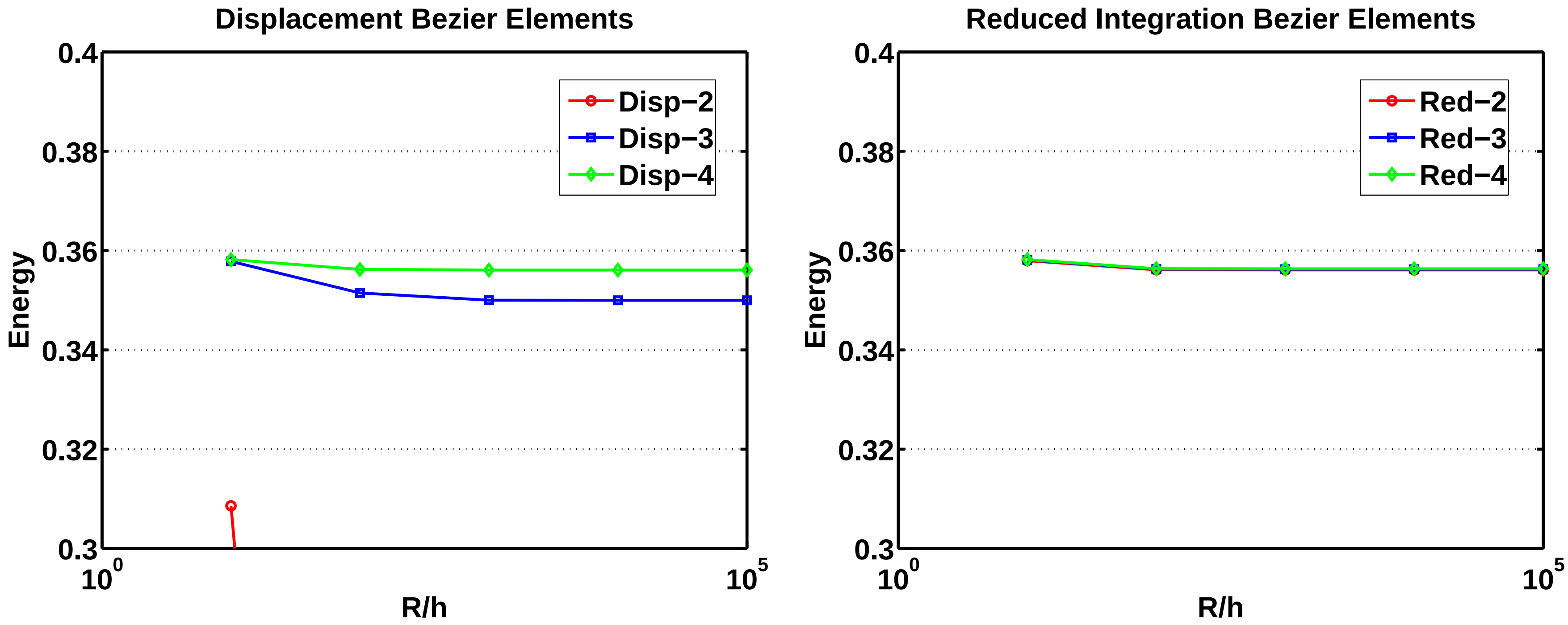
- End shear reduced integration forces: 4-elements





Isogometric Analysis of Structures

- **Uniform load reduced integration results: 4-elements**



- Reduced integration good for 1-d problems.
  Need more general form for multi-dimensions.

Isogometric Analysis of Structures

- **Discrete Strain Gap:**
  - Method for constraints. Lowers order of interpolation. (Bletzinger *et al.*:2000; Koschnick *et al.*: 2005)
  - Replaces locking terms by derivative form of interpolations.
  - Example: Shear strain in straight Timoshenko beam.

$$\gamma = \frac{\mathrm{d}w}{\mathrm{d}x} + \varphi = \sum_b \left(\frac{\mathrm{d}N^b}{\mathrm{d}x}\, \tilde{w}_b + N^b\, \tilde{\varphi}_b \right) \rightarrow \boxed{\bar{\gamma} = \sum_a \frac{\mathrm{d}N^b}{\mathrm{d}x}\, \tilde{\gamma}_a}$$

where

$$\tilde{\gamma}_a = \sum_b \int_{-1}^{\xi_a} \left(\frac{\mathrm{d}N^b}{\mathrm{d}x}\, \tilde{w}_b + N^b\, \tilde{\varphi}_b \right) j(\xi)\, \mathrm{d}\xi$$

with $j(\xi) = \mathrm{d}x/\mathrm{d}\xi$ and ξ_a nodal coordinates.

- Used in above references for solids, beams, plates & shells.

Isogometric Analysis of Structures: DSG

- Discrete strain gap (DSG) used with NURBS for beams by Echter & Bischoff (CMAME, vol 199, pp 374ff, 2010).

- Using NURBS basis led to tangent with fully occupied shear terms.

- Using Bézier extraction offers possibility of local approximations.
  - Disadvantage: Continuity between elements only C^0.
  - Disadvantage: Many more nodal points.
  - Advantage: Offers possible scheme for all C^0 problems.

Isogometric Analysis of Structures

- Interpolate DSG shear on Bézier element.
- Given Bernstein interpolations: $-1 \leq \xi \leq 1$

$$
\begin{aligned}
N_1^B &= \frac{1}{2^n}(1-\xi)^n \\
N_2^B &= \frac{n}{2^n}(1-\xi)^{n-1}(1+\xi) \\
\cdots \\
N_n^B &= \frac{1}{2^n}(1+\xi)^n
\end{aligned}
$$

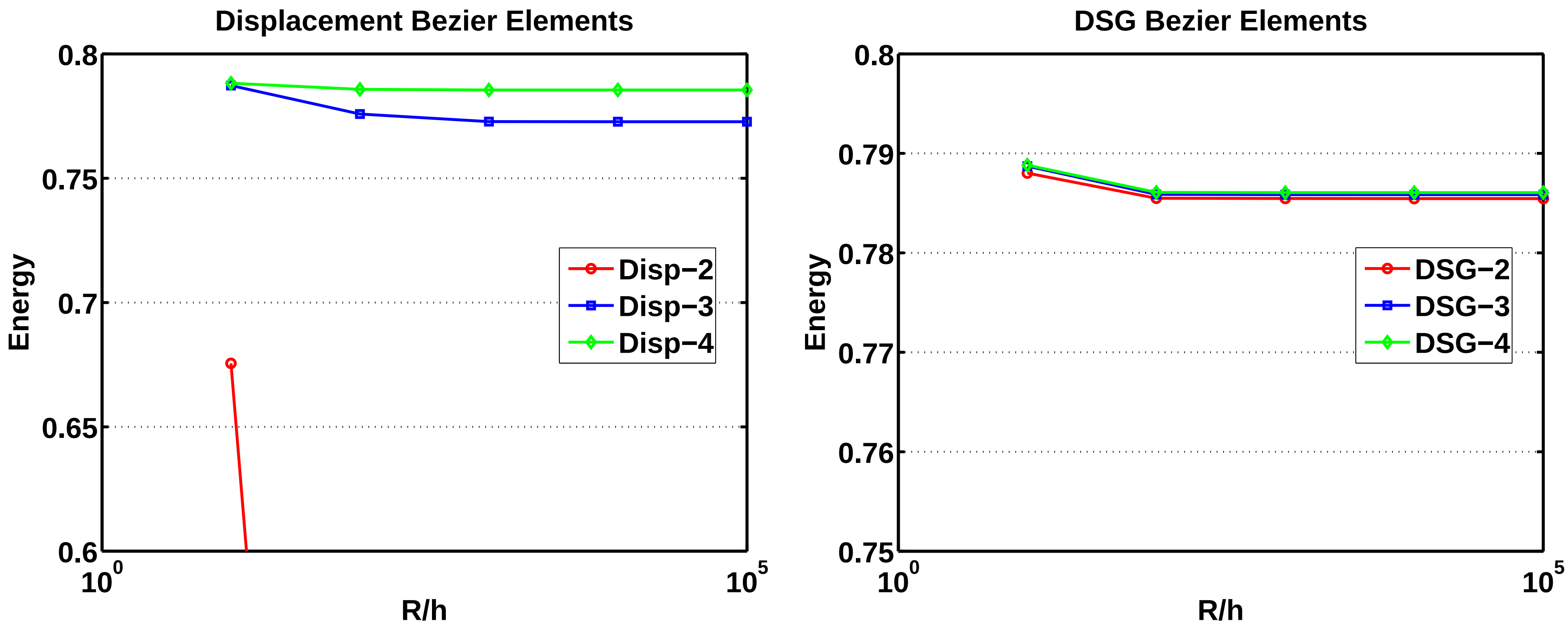
- Transform from Bernstein to Lagrange interpolation

$$
\begin{Bmatrix} N_1^B(\xi) \\ N_2^B(\xi) \\ \cdots \\ N_n^B(\xi) \end{Bmatrix} = \begin{bmatrix} N_1^B(\xi_1) & N_1^B(\xi_2) & \ldots & N_1^B(\xi_n) \\ N_2^B(\xi_1) & N_2^B(\xi_2) & \ldots & N_2^B(\xi_n) \\ \vdots & \vdots & \ddots & \vdots \\ N_n^B(\xi_1) & N_n^B(\xi_2) & \ldots & N_n^B(\xi_n) \end{bmatrix} \begin{Bmatrix} N_1^L(\xi) \\ N_2^L(\xi) \\ \cdots \\ N_n^L(\xi) \end{Bmatrix}
$$

where ξ_a are positions of Lagrange nodes.

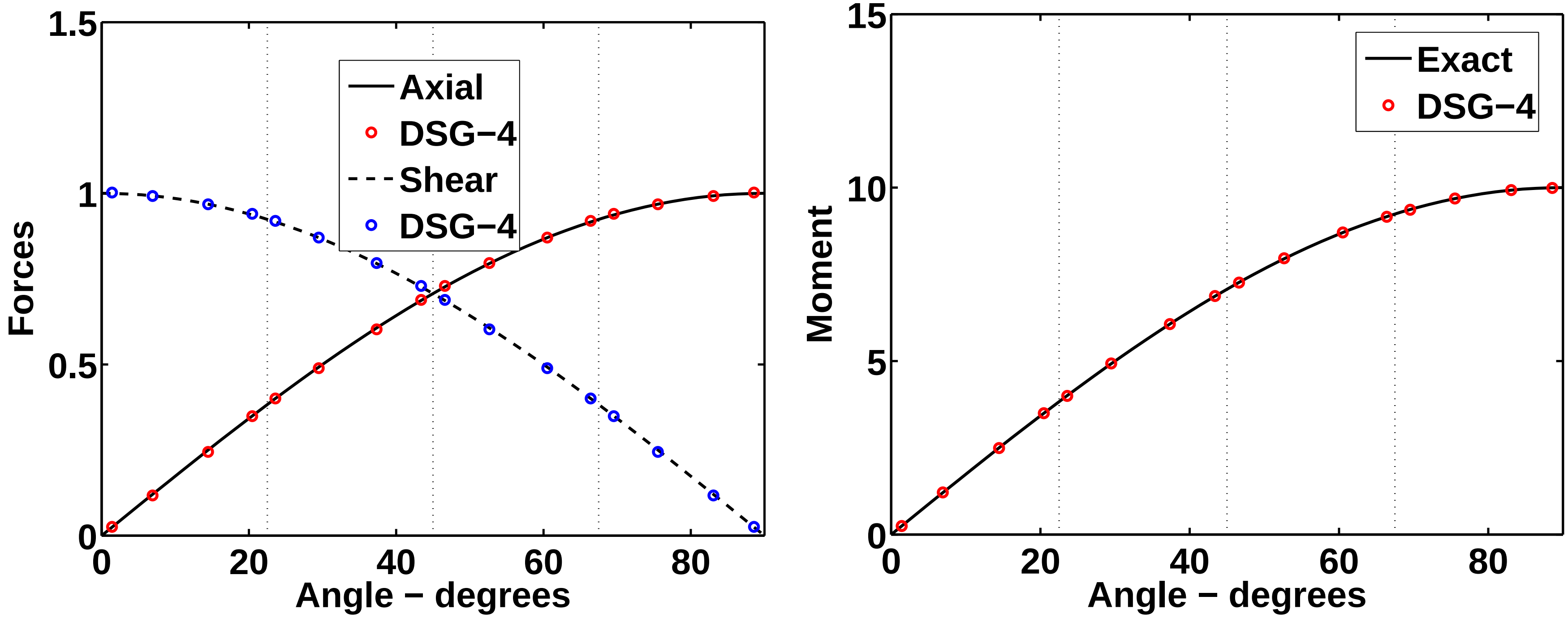
Isogometric Analysis of Structures

- End shear DSG results: 4-elements



Isogometric Analysis of Structures

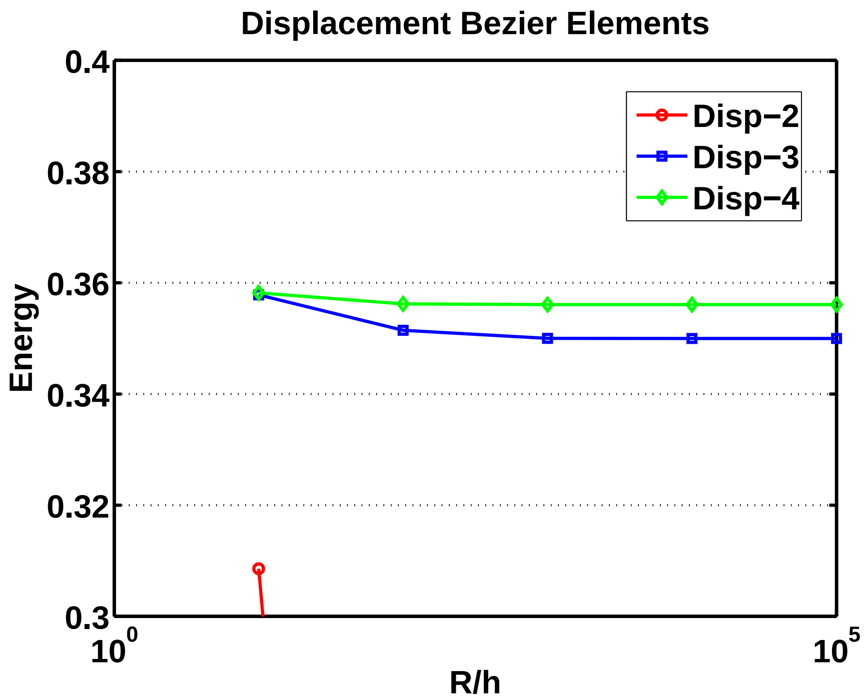
- **End shear DSG force results: 4-elements, $h = 0.01$**

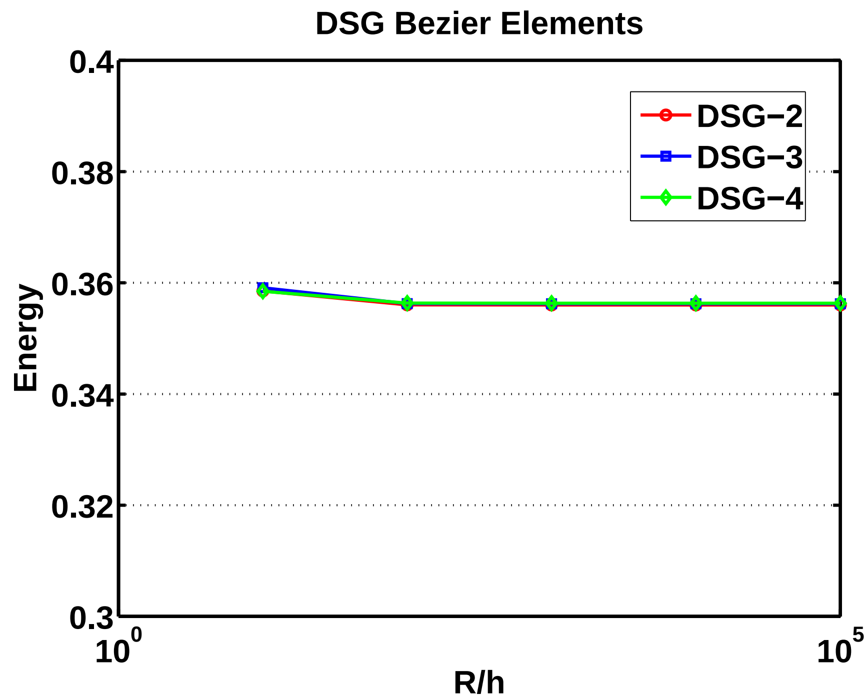


- DSG expensive compared to reduced integration
  Provides method that works for multi-dimensions

Isogometric Analysis of Structures

- **Uniform load DSG results: 4-elements**





Isogometric Analysis of Structures

Closure:

- Described aspects of implementation of IGA into FE software.
- Considered Bézier extraction form for element development.
- Considered issue of locking in structural elements.
- For 1-d Bézier elements reduced integration adequate.
- DSG offers possibility of treatment in multi-dimensions.
- Displacement form adequate for large p!