Isogeometric Analysis of Solids & Structures
Small and Finite Deformation Applications

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Isogeometric Analysis of Solids & Structures

Outline of presentation:

- Overview of FEAP for IGA
  - NURBS meshing: Control points, knots, sides
  - IGA capable elements
  - k-refinement: degree elevation & knot insertion
  - Coding summary

- Bézier extraction form for developments. T-spline inputs.

- Applications.
  - 1-d rod: frequencies and mode shapes
  - Finite deformation of solids: Near incompressibility
  - Thin shells
  - 2-d solutions using T-splines: Graphics construction
  - Composide plate: Integration of shell and solid

- Closure
Isogometric Analysis of Solids & Structures

• Acknowledgements:

  – Tom Hughes (UT, Austin) & Yuri Bazilevs (UC, San Diego): Reports & initial small program for isogeometric analysis.

  – Sanjay Govindjee & Toby Mitchell (UC, Berkeley): Local form for essential boundary conditions & plots.

  – Michael A. Scott (UT, Austin): T-splines meshes.
Brief Overview of FEAP

• **FEAP** - Finite Element Analysis Program.

• Research and educational software package developed at University of California, Berkeley.

• Includes element library: Solids, Thermal, Frames, Plates, Membranes & Shells.

• Elements for both small and large deformation analysis.

• Material library for: Elastic, visco-elastic, elasto-plastic, ...

• Solution algorithms by command language statements.

• Screen and hard copy plotting options.

• User module interfaces for elements, meshing, solution, plots.
  – Used for NURBS and T-spline isogeometric solutions.
Isogeometric Modeling in FEAP

- Isogeometric models described with ‘user modules’ for:
  - Knot vectors (open)
  - Control points and weights
  - Tensor product NURBS (Non-Uniform Rational B-Splines)
  - T-splines (mostly for surfaces)

- Current IGA elements:
  - Displacement formulations (all FEAP solid elements)
  - Mixed $u-p-\theta$ formulation (2-d & 3-d finite deformation)
  - Thermal elements (2-d & 3-d)
  - ‘User module’ elements for rod, beam, plate & shell.

- Boundary conditions (only restricted types, no contact)

- Most solution options work (transient, eigenpairs, etc.)

- Plot of mesh, displacement & stress contours, eigen-pairs, etc.
Isogeometric Modeling with NURBS

**B-Spline interpolations**

- **Open knot vector:**
  
  Length $m$ values.
  
  $\Xi = \left[ \frac{\xi_1, \ldots, \xi_1, \xi_2, \ldots, \xi_r, \ldots, \xi_r}{p+1 \text{ times}}, \frac{\xi_1, \ldots, \xi_1, \xi_2, \ldots, \xi_r, \ldots, \xi_r}{p+1 \text{ times}} \right]$

- **Start basis:**
  
  $B_{i,0} = \begin{cases} 
  1; & \text{if } \xi_i \leq \xi < \xi_{i+1} \\
  0; & \text{otherwise}
  \end{cases}$

- **Recursion $n$ functions:**
  
  $B_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i}B_{i,p-1}(\xi) + \frac{\xi_i + p + 1 - \xi}{\xi_{i+p+1} - \xi_{i+1}}B_{i+1,p-1}(\xi)$

  where $n = m - p - 1$ for open knots.
  
  e.g., $\Xi = \left[ 0, 0, 0, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, 1, 1, 1 \right]$, $p = 3$ for figure.
Isogeometric Modeling with NURBS

• Non-Uniform Rational B-spline (NURBS) defined by

\[ N_{i,p}(\xi) = \frac{B_{i,p}(\xi) w_i}{\sum_{j=1}^{n} B_{j,p}(\xi) w_j} \]

where \( w_i \) are set of \( n \) weights defining shape of NURBS.

• Appropriate values of \( w_i \) permit description different types of curves; e.g., conic surfaces in addition to polynomials
  – Polynomials in \( \xi \): Weights \( w_i = 1 \) yields: \( N_{i,p}(\xi) = B_{i,p}(\xi) \)
  – Circular arc: \( x = N_{1,2}(\xi) \tilde{x}_1 + N_{2,2}(\xi) \tilde{x}_2 + N_{3,2}(\xi) \tilde{x}_3 \)

• Example: \( \theta = 60^\circ \) circular arc with radius 5
  \( x_1 = (5, 0) \)
  \( x_2 = (5, \tan(\theta/2)) \)
  \( x_3 = (5, \cos(\theta), 5 \sin(\theta)) \)
  \( (w_1, w_2, w_3) = (1, \cos(\theta/2), 1) \)
Isogeometric Modeling with NURBS

Analysis procedure in FEAP

• Define coarse set of control points, knots, 1-d knot-point list, side-patch description:
• Example: Curved beam – input NURBS, knot, side & block

NURBs
1 0 10.0 0.0 1.0
2 0 10.0 10.0 1.0/sqrt(2)
3 0 0.0 10.0 1.0

KNOT ! (Quadratic)
knot 1 0.0 0.0 0.0 1.0 1.0 1.0

NSIDe
side 1 0 1 1 2 3

NBLOck
block 1 1 1
Isogeometric Modeling with NURBS

- Need to add material properties, loading and boundary conditions. Use standard *FEAP* commands for most.

- Analysis requires degree elevation and knot insertion.

- Example: Elevate order to cubic interpolation

  BATCh
  ELEVate KNOT 1 1
  END

- Knot vector now cubic: \( \Xi = (0, 0, 0, 0, 1, 1, 1, 1) \)
Isogeometric Modeling with NURBS

Add knots: $k$-refinement in circumferential direction

(a) Insert knot 1
(b) Insert knot 2
(c) Insert knot 3: $\Xi = (0, 0, 0, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, 1, 1, 1)$
Isogeometric Modeling with NURBS

Add knots: \( k \)-refinement in FEAP

- Knot insertion performed as
  
  BATCh
  
  INSERT KNOT knum uu rr
  
  END

  where \( uu \) knot value & \( rr \) number times to repeat.

- Each knot insertion lowers continuity by one order.

- Degree elevation and knot insertion create mesh for analysis.

- **Elements** defined on knot intervals.
Isogeometric Modeling

\textbf{\textit{k-refinement in FEAP (Cont.)}}

- Each elevation or insertion creates a flat mesh \texttt{NURB_mesh}
- Can be used to describe problems recursively in input file using \texttt{INCLude NURB_mesh}
- Use loops (in input file) to perform repeated insertions
  
  \begin{verbatim}
  LOOP,3
    PARAMeter
    d = d + 0.0.25
    INCLude Ixxxx ! (Contains INCLude NURB_MESH)
  BATCh
    INSErt KNOT 1 d 1
  END
  NEXT
  \end{verbatim}

  inserts knot 1 three times at intervals of 0.25 units.
Isogeometric Element Development

- Typical displacement formulation element module

```plaintext
Call Quadrature()
Loop: L = 1, LINT
  Call Interp()
  Call Constitutive()
  Form Resid
  Form Tangent
End Loop
```

- IGA modifies **quadrature** and **interpolation** modules.
Isogometric Element Development

- Some implemented IGA user elements:

<table>
<thead>
<tr>
<th>ELMT</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>NURBS Euler-Bernoulli frame</td>
</tr>
<tr>
<td>2</td>
<td>NURBS 1-d rod.</td>
</tr>
<tr>
<td>5</td>
<td>NURBS &amp; T-spline thin $C^1$ plate</td>
</tr>
<tr>
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<td>NURBS &amp; T-spline thin membrane</td>
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<td>Thin non-linear shell (Kiendl et al.)</td>
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<tr>
<td>25</td>
<td>Thin linear shell (Kiendl et al.)</td>
</tr>
<tr>
<td>43</td>
<td>Linear curved beam with shear deformation.</td>
</tr>
</tbody>
</table>

FEAP user elements for NURBS/T-spline solutions.
Isogeometric Solutions in FEAP

- Isogeometric developments use NURBS blocks and/or T-splines.

- Requires lots of coding: Degree elevation, knot insertion, etc.
  - References: Piegl & Tiller, 1997; Hughes et al. papers/codes.

- Graphics currently mapped to 4-node quadrilaterals.
Isogeometric Solutions in FEAP

- T-splines greatly simplifies coding requirements:
  Key idea is use of extraction operator to define shape functions.

- Extension to T-splines (required to be analysis suitable):
  - T-spline basis function defined from local tensor product parameter domain.
  - Bézier extraction defines row in each element within support of a basis function.

- See: [http://www.ices.utexas.edu/research/reports](http://www.ices.utexas.edu/research/reports)
Bézier Extraction Form for Elements

- Extraction converts B-splines & NURBS to Bézier form.
  - Example: Cubic B-Spline (elements between dotted lines)

After repeated knot insertion obtain Bézier basis (right figure):

\[ N^e(\xi) = C^eB(\xi) \] where \( B(\xi) \) are Bernstein polynomials

- Shape functions: extraction operator times Bernstein polynomials.
Bézier Extraction Form for Elements

- For curves: Interpolations in rational form
  \[ R_a = \frac{w_a B_a(\xi)}{W(\xi)} \quad \text{where} \quad W(\xi) = \sum_b w_b B_b(\xi) \]
  where \( w_a \) is a weight for the \( a \) basis function.

- Permits representation of conics and other curves:
  \( w_a = 1 \) gives polynomial.

- Surfaces and solids use tensor products of \( R_a \) functions.

- 2-d & 3-d NURBS to extraction operator form becomes:
  \[ N^e(\xi) = C^e R^e(\xi) \]


- Incorporate into shape function modules: FE, NURBS, T-splines
Bézier Extraction Form for Elements

• Coordinate interpolation with extraction operator:

\[
x(\xi) = \sum_{b=1}^{m} \tilde{x}_b R_b(\xi) = \sum_{b=1}^{m} \sum_{a=1}^{n} \tilde{x}_a C_{ab} R_b(\xi) = \sum_{a=1}^{n} \tilde{x}_a \sum_{b=1}^{m} C_{ab} R_b(\xi) = \sum_{a=1}^{n} \tilde{x}_a N_a(\xi)
\]

where \( m \) number of Bézier points and \( n \) number of control points.

• Isoparametric form for other variables:

\[
u(\xi) = \sum_{b=1}^{m} \tilde{u}_b R_b(\xi) = \sum_{a=1}^{n} \sum_{b=1}^{m} \tilde{u}_a C_{ab} R_b(\xi) = \sum_{a=1}^{n} \tilde{u}_a N_a(\xi)
\]

• Permits element development with either NURBS or Bézier form.
Isogometric Analysis of Solids & Structures

Applications
Isogometric Analysis of Solids & Structures

- Theory for rod:

\[ EA \frac{\partial^2 u}{\partial x^2} - \rho A \frac{\partial^2 u}{\partial t^2} = q \quad ; \quad u = 0 \text{ at } x = 0, L \]

- Natural frequencies: \( \omega_n^2 = \frac{n^2 \pi^2 E}{\rho L^2} \)

- Example: \( L = 10, \ E = 10, \ \rho = 0.1, \ A = 1: \ \omega_n = \frac{1}{2} n \text{ Hz.} \)
  
  Mode shapes: \( \phi(x) = \sqrt{2} \sin \frac{n \pi x}{10} \) (Mass orthonormal)

- Solve using equal size Lagrange and NURBS elements with:
  - Consistent mass
  - Row sum lumped mass
Frequency error: (Based on O(100) elements)

Solid lines: Consistent mass
Dotted lines: Lumped mass - row sum

Reference: See also book by Cottrell et al. (2009)
• Mode shapes: Consistent mass cubic NURBS
  10 knot intervals: 11 modes

Very good representation for nearly all modes.
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- Mode shapes: Row sum lumped mass cubic NURBS
- 10 knot intervals: 11 modes

Note: Order of eigenvalues not in correct sequence!
Isogometric Analysis of Solids & Structures

- Mode shapes: Row sum lumped mass cubic Lagrange
- 4 elements: 11 modes

Note $C^0$ structure for $n > 4!$
Isogeometric Modeling of Solids by NURBS

- Solution of problems in 2-d & 3-d solid mechanics

- Small and large deformation problems considered

- Geometry described by tensor product NURBS on patches. Boundaries between patches only \( C_0 \) continuous (match control points).

- Approximations in 2-d taken as

\[
X = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} N_{i,p}(\xi) N_{j,q}(\eta) \tilde{X}_{ij} \quad \text{and} \quad u = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} N_{i,p}(\xi) N_{j,q}(\eta) \tilde{u}_{ij}
\]

where \( X \) are reference coordinates and \( u \) displacements. 3-d forms add third interpolation \( N_{k,r}(\zeta) \).

- Strains, deformation gradients, etc. defined in usual isoparametric method.
Isogeometric Modeling of Solids by NURBS

- Problems with near incompressibility can use mixed $u$-$p$-$\theta$ form

$$\Pi(u, p, \theta) = \int_\Omega W(\bar{C}) \, d\Omega + \int_\Omega p(J - \theta) \, d\Omega + \Pi_{\text{ext}}$$

- Split $F$ into volumetric and isochoric parts

$$F = F_{\text{vol}} F_{\text{iso}} \quad \text{with} \quad \det F = J$$

where $F_{\text{vol}} = J^{1/3} \mathbf{1}$ and $F_{\text{iso}} = J^{-1/3} F$

Note: $\det F_{\text{vol}} = J$ and $\det F_{\text{iso}} = 1$.

- Mixed form replaces $J$ by $\theta$ in volumetric part

$$\bar{F} = \left(\frac{\theta}{J}\right)^{1/3} F \quad \text{with} \quad \det \bar{F} = \theta$$

- Yields mixed right Cauchy-Green deformation tensor

$$\bar{C} = \bar{F}^T \bar{F}$$

used to define material models.
Isogeometric Modeling of Solids by NURBS

• First variation of functional

\[ \delta \Pi = \int_{\Omega} \delta \bar{C} : \frac{1}{2} \bar{S} \, d\Omega + \int_{\Omega} \delta p (J - \theta) \, d\Omega + \int_{\Omega} p (\delta J - \delta \theta) \, d\Omega - \delta \Pi_{ext} = 0 \]

where \( \bar{S} = 2 \frac{\partial W}{\partial \bar{C}} \).

• Approximate solutions

\[ X = \sum_a N_a(\xi) \bar{X}_a \quad \text{and} \quad u = \sum_a N_a(\xi) \bar{u}_a(t) \]

\[ \theta = \sum_b L_b(\xi) \bar{\theta}_b = L \tilde{\theta} \quad \text{and} \quad p = \sum_c M_c(\xi) \bar{p}_c = M \tilde{p} \]

where \( N_a, L_b, M_b \) denote appropriate shape functions (e.g., NURBS, Lagrange, polynomial, etc.)
Isogeometric Modeling of Solids by NURBS

- Algebraic problem for Newton solution: Alternative solutions

\[
\begin{bmatrix}
K_{uu} & K_{u\theta} & K_{up} \\
K_{\theta u} & K_{\theta\theta} & K_{\theta p} \\
K_{pu} & K_{p\theta} & 0
\end{bmatrix}
\begin{bmatrix}
d\ddot{u} \\
d\ddot{\theta} \\
d\ddot{p}
\end{bmatrix}
= 
\begin{bmatrix}
R_u \\
R_\theta \\
R_p
\end{bmatrix}
\]

- Residuals expressed as

\[
\begin{align*}
R_u &= f - \int_\omega B^T \hat{\sigma} \, d\omega \\
R_\theta &= -\int_\Omega L^T (\bar{p} - p) \, d\Omega \\
R_p &= -\int_\Omega M^T (J - \theta) \, d\Omega
\end{align*}
\]

where

\[
\hat{\sigma} = \bar{\sigma} + m (p - \bar{p}) \quad \text{with} \quad \bar{p} = \frac{1}{3} m^T \bar{\sigma}
\]
Isogeometric Modeling of Solids by NURBS

Continuous \( p-\theta \) approximations

- Usual discontinuous approximation for \( p \) and \( \theta \) does not work for NURBS

- \( \tilde{F} \) and \( \tilde{B} \) proposed by Elguedj et al. (CMAME, 197, 2008)
  - Uses continuous \( u-p \) NURBS method.
  - \( p \) NURBS one order lower than \( u \) order.
  - Requires special solution scheme.

- Continuous \( u-p-\theta \) mixed approach can be solved by standard approach.

- Continuous \( u-p \) discontinuous \( \theta \) mixed approach also can be solved by standard approach.

- Ref: RLT, IJNME, 87, 2011.
Isogeometric Modeling of Solids by NURBS

Elastic-Plastic Necking: 2-d Example

• Plane strain strip under uniform extension

• Finite strain elastic-plastic model with $J_2$ yield function.

• Material properties are:
  \[ K = 164.206 \ ; \ G = 80.1938 \ ; \ \sigma_0 = 0.45 \ ; \ \sigma_\infty = 0.715 \ ; \ h = 0.12924 \ ; \]
  Uniaxial yield stress given by
  \[ \sigma_y = \sigma_\infty + (\sigma_0 - \sigma_\infty) \exp(-\beta e_p) + \sqrt{\frac{2}{3}} h e_p \]
  where $e_p$ effective plastic strain and $h$ isotropic hardening.

• Geometry: Length is 53.334 and width is 12.826. Symmetry used for one quadrant model (reduce center to 0.982 of width).

• Use quadratic NURBS with 10 knots in width and 20 in length. Analysis also performed with Q1/P0 & Q2/P1 elements.
Isogeometric Modeling of Solids by NURBS

Elastic-Plastic Necking Example

(a) Mesh
(b) Mises stress at full extension
Isogeometric Modeling

Elastic-Plastic Necking Example

(c) Necking displacement  (d) Load-displacement
Isogometric Analysis of Solids & Structures

Torsion of block: 3-d Example

- Axial displacement $4 \times 4 \times 16$ mesh

![Axial displacement diagram]

(a) Displacement $u_3 = \pm 0.0121$

(b) Mixed $u - p - \theta$

$u_3 = \pm 0.0117$

- Loaded by rotating top by $90^\circ$. 

![Mixed diagram]
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Torsion of block: 3-d Example

• Shear Stress: $\tau_{13}$ $4 \times 4 \times 16$ mesh

(a) Displacement

$\tau_{31} = \pm 12.0$

(b) Mixed $u - p - \theta$

$\tau_{31} = \pm 12.1$

• Results consistent
Isogometric Analysis of Solids & Structures

Torsion of block: 3-d Example

• Axial Stress: \( \sigma_3 \) 4 \( \times \) 4 \( \times \) 16 mesh

- \( \sigma_3 = -48.1 \) to 45.1
- \( \sigma_3 = -5.12 \) to 5.75

- Shows locking effect in quadratic NURBS displacement model.
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Torsion of block: 3-d Example

- Axial displacement: Cubic NURBS $8 \times 8 \times 64$ mesh

Full quadrature (4x4). Reduced quadrature (3x3).

- Loaded by rotating top by $90^\circ$. Increments $30^\circ$. 
Isogometric Analysis of Solids & Structures

Torsion of block: 3-d Example

- Axial Stress: $\sigma_3$ using cubic NURBS $8 \times 8 \times 64$ mesh

Full quadrature (4x4). Reduced quadrature (3x3).

- Loaded by rotating top by $90^\circ$. Increments $30^\circ$. 
Isogometric Analysis of Structures

- Thin Shell: Rotation free Kirchhoff-Love
  (Kiendl, et al., CMAME, 198, 2009)

- NURBS or T-spline interpolation on reference surface.
  Currently limited to single NURBS patch (no bending strip).

- Implemented for linear and non-linear kinematics.

- Constitution: Isotropic St. Venant-Kirchhoff

- Slope boundary by penalty or link of control points.
**Isogometric Analysis of Structures**

- **Curved beam example:**
  Results for 20-cubic elements with end shear.

- Roundoff high for $R/h > 1000$
Isogometric Analysis of Solids & Structures

- Scordelis-Lo Barrel Vault:

\[ E = 3 \times 10^3 \text{ kN/m}^2 \]
\[ v = 0 \]
\[ g = 0.09 \text{ kN/m}^2 \]

\[ t = 3 \text{ in} \]
\[ R = 25 \text{ ft} \]
\[ 40^\circ \]

Supported by rigid diaphragm
\[ u = 0 \]
\[ w = 0 \]

Free edge

**DISPLACEMENT 2**

**FORCE -11**

**MOMENT 11**
Isogometric Analysis of Structures

Roll-up Problem:

- Problem data: \( L = 10, \quad E = 10.92 \times 10^6, \quad \nu = 0.3, \quad h = 0.1 \)
- Solve using 20 knot intervals using \( 3^{rd} \) and \( 4^{th} \) degree NURBS
Isogometric Analysis of Structures

Roll-up Results:

- Final shape not a circle since NURBS weights unity.

- Error small for 4\textsuperscript{th} degree NURBS: $R = 1.592 \pm 0.0002$
Solving Problems with T-spline Meshes

- Bézier extraction operator form allows creation of shape functions in same way as for isoparametric elements with Lagrange interpolation.

- Remaining problem is creating mesh in T-spline form and computing all extraction operators.

- Current work based on output from T-Splines, Inc. software: [http://www.tsplines.com](http://www.tsplines.com) with subsequent refinement (Scott, 2011).

- Output provides: Control points, element connections & extraction operators. (No boundary conditions, loads, properties, etc.)

- Restricted to SURFACES and VOLUMES (surfaces with thickness).
Solving Problems with T-spline Meshes

- Basic *FEAP* input file:

  FEAP * * Title information
  0 0 0 0 0 0 ! Control record all zeros

  MATERial 1
  SOLId <MEMBrane, THERmal>
  ......

  T-SPline
  MATE number 1
  PLOT interval 2 ! Number divisions knot interval
  FILE filename.ext ! Name of refinement file

  Boundary conditions loading etc.

END
Solving Problems with T-spline Meshes

- Refinement file structure (`filename.ext`): M.A. Scott

```
SURFACE <VOLUME>
dim 2  <3>
deg 3 3  <3>
funcs 595  ! Number of control points
elems 160  ! Number of elements
g0  x0  y0  z0  w0
g1  x1  y1  z1  w1

.......!
elem0 16 3 3  <64 3 3 3>! 16 <64> = nel, 3 = order (p)
c1  c2  .....  c16  ! Control point numbers

....... ! Extraction operator (C\_e)
elem1

.......
end
```
Solving Problems with T-spline Meshes

- Example T-Mesh: 119 Control points; 80 elements.

Dimensions: $a = 5$; $b = 10$; $u(x, 0) = 0.1$; $E = 10,920$ & $\nu = 0.3$. 
Plotting NURBS & T-spline Results

- *FEAP* plots surfaces of all element types as 3-node triangles or 4-node quadrilaterals.

- Results from NURBS & T-splines converted to plot form by:
  - Project stress component $\sigma$ on Bézier elements:
    \[ \hat{x}^e = \bar{x}^e C^e \quad \text{(Bezier mesh nodes)} \]
    \[ \sigma(\xi) = \sum_{b=1}^{m} \hat{\sigma}_b R_b(\xi) \]
  - (a) Do discrete least squares on each element:
    \[ \sum_{l=1}^{\text{lint}} \left[ \sigma(\xi_l) - \sigma_l \right]^2 j(\xi_l) W_l = \min \]
    
    Use partial diagonal matrix allows reduced quadrature.
  - (b) Average $\hat{\sigma}_b$ at Bézier nodes (Mitchell, et al., 2011).
  - Divide Bézier element into 4-node quads & compute nodal $\sigma$. 
Plotting T-spline Results

- Bézier mesh

- 787 Bézier mesh nodes vs. 119 T-spline mesh nodes!
Plotting T-spline Results

- Divide Bézier mesh into sub-elements:
  - Plot option for T-spline files

```plaintext
T-SPline
MATEerial number ma
PLOT interval p_int
FILE filename.ext
```
Plotting T-spline Results

- Plot of $\sigma_{11}$ for 7 subdivisions of Bézier element.
Plotting T-spline Results

- Method works for material models with internal variables (e.g., viscoelasticity or plasticity)

- Least squares on cubic Bézier elements requires $l_{int}$ be 16 for surface elements and 64 for volume elements. Modified form uses 9 and 27!

- Use of least squares on T-spline elements directly fails when number of element control points exceeds number of independent shape functions.

- Above method also used to imposed essential boundary conditions on NURBS meshes (Mitchell, et al., PW Volume, Springer, 2011).

- An alternative is superconvergent patch recovery (ZZ-projection).
Isogometric Analysis of Solids & Structures

Bumper frequency and mode shapes:

- Properties: $E = 10.92 \times 10^6$, $\nu = 0.3$, $\rho = 8.12$, $h = 0.5$
- T-spline mesh: 705 Control points, 508 elements.
Bumper frequency and mode shapes:

- Properties: $E = 10.92 \times 10^6$, $\nu = 0.3$, $\rho = 8.12$, $h = 0.5$
- T-spline mesh: 705 Control points, 508 elements.

(a) Mode 7: $\omega = 0.724$Hz.

- First deformation mode: Bending
Isogometric Analysis of Solids & Structures

Bumper frequency and mode shapes:

- Properties: $E = 10.92 \times 10^6$, $\nu = 0.3$, $\rho = 8.12$, $h = 0.5$
  T-spline mesh: 705 Control points, 508 elements.

(b) Mode 8: $\omega = 1.606$Hz.

- Second deformation mode: Torsion
Bumper frequency and mode shapes:

- Properties: $E = 10.92 \times 10^6$, $\nu = 0.3$, $\rho = 8.12$, $h = 0.5$
- T-spline mesh: 705 Control points, 508 elements.

(c) Mode 10: $\omega = 2.977$Hz.

- Second deformation mode: Second bending
Isogometric Analysis of Solids & Structures

- **Composite Plate: Combined shell & solid:**

  - Faces modelled by thin shell
    \[ E = 10.92 \times 10^6, \quad \nu = 0.3, \quad h = 0.01 \]
  
  - Core modelled by elastic solid
    \[ E = 2000, \quad \nu = 0.0 \]
  
  - Uniform load on top of plate
    \[ q = -100 \]
  
  - Simply supported on bottom edge.
  
  - Symmetry imposed (penalty)
  
  - T-spline Mesh: 748 Control points, 448 elements.
Isogometric Analysis of Solids & Structures

- Combined shell & solid:

Assemble: Shell (faces) & Solid (core)
Isogometric Analysis of Solids & Structures

- Combined shell & solid: Deformed shape \( \times 10 \)

(a) Top View: \( u_3 \)
(b) Bottom View: \( u_3 \)
Small Deformation Solution
**Isogometric Analysis of Solids & Structures**

- **Combined shell & solid: Deformed shape \times 10**

(a) Top View: $u_3$  
(b) Bottom View: $u_3$  

Finite Deformation Solution
Isogometric Analysis of Solids & Structures

Closure:

• Described methods used to implement IGA into FEAP

• Current capabilities restricted to NURBS blocks and T-spline files from M.A. Scott. To be released by T-Spline soon?

• Applications:
  – Examples show good results for most problems solved
  – Basic elements provide basis for several general classes of solid, structure and/or thermal problems.

• Future efforts & needs:
  – Contact and other boundary conditions.
  – Efficient and more accurate mass treatment for explicit codes.
  – Methods to treat volume/shear constraints for T-splines.
  – General 3-d meshing.
  – Etc.